ON A NEW APPLICATION OF JACOBI POLYNOMIALS IN CONNECTION WITH THE MEAN VALUE THEOREM

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Let us consider the classical theorem of mean value, which states that

$$U(\xi) = \frac{1}{b-a}\int_a^b U(x)dx \tag{1}$$

or

$$F'(\xi) = \frac{1}{b-a}\left[F(b) - F(a)\right] \tag{1'}$$

can be satisfied by at least one value of ξ inside[1] the interval (a, b).

In the general case we can add no further precision concerning the position of the value ξ inside the interval (a, b). But if we consider only functions $U(x)$ belonging to a definite class of functions, we can, sometimes, give a more precise determination for this value ξ. We can, in particular, for some classes of functions, determine intervals (a', b'), concentric to (a, b), with

$$b' - a' = \theta(b - a), \qquad 0 \leqq \theta \leqq 1, \tag{2}$$

and such that (1) holds for at least one value ξ inside (a', b'), for every function $U(x)$ belonging to the class considered and for every interval (a, b) for which the classical mean value theorem holds. The smallest number θ which has the above mentioned property for a given class of functions is called its "contraction factor." It results from this definition that the value of the contraction factor depends only on the class of functions considered and is independent of all other factors, such as the interval (a, b), and so on.

If we replace the equation (1) by (1') and repeat the foregoing literally, we define in exactly the same way the contraction factors for classes of functions $F(x)$.

The existence of a contraction factor for certain classes of functions, particularly for polynomials of a real variable, has been proved by Paul Montel.[2] The value of θ as a function of the degree n of the polynomials considered was found independently and almost at the

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[1] The expression "inside" in this paper means: within or at the ends.

[2] P. Montel, Bull. Soc. Math. France vol. 58 (1930) pp. 105–126. See also D. Pompeiu, Annales Scientifiques de l'Université de Jassy vol. 15 (1929) p. 335.

same time by Tchakaloff[3] and Biernacki.[4] Their results have been generalized since by many authors, among whom we quote Favard,[5] Anghelutza,[6] H. L. Krall[7] and Cioranescu,[8] whose paper is particularly important for us.

Let $U(y_1, y_2, \cdots, y_\nu)$ be an analytical function of ν independent variables, polyharmonical of the order $2m$ and of mean value zero within the hypersphere Σ_ν of its ν-dimensional space. Therefore

$$\Delta^{(2m)}U = 0, \tag{3}$$

$$\int_{\Sigma_\nu} U d\tau = 0, \tag{4}$$

where $\Delta^{(2m)}$ is the operator of Laplace applied successively $2m$ times and the integral is extended over the inside of the hypersphere Σ_ν of volume element $d\tau$.

We consider now the system of $2m$ equations with the $2m$ unknowns k_i, x_i:

$$\sum_{i=1}^{m} k_i x_i^{2q} = \frac{\nu}{\nu + 2q}, \qquad q = 0, 1, 2, \cdots, 2m-1, \tag{5}$$

and let x_M be the largest of the solutions x_i of this system. N. Cioranescu proved in his above mentioned paper that x_M is the contraction factor of $U(y_1, y_2, \cdots, y_\nu)$ if:

(a) all x_i are real and inside $(-1, +1)$;

(b) all k_i are positive.

We remember that this statement means: If (3), (4), (a) and (b) are satisfied and R is the radius of Σ_ν, there is at least one point P_0 inside the hypersphere of radius $x_M R$ which makes $U(P_0) = 0$.

From Cioranescu's demonstration it follows also that if (a) and (b) are satisfied, but the order of U is odd, so that

$$\Delta^{(2m+1)}U = 0, \tag{3'}$$

$$\int_{\Sigma_\nu} U d\tau = 0, \tag{4'}$$

[3] L. Tchakaloff, C. R. Acad. Sci. Paris vol. 192 (1931) p. 32.

[4] M. Biernacki, Bulletin de Mathématiques et de Physique, pures et appliquées de l'École Polytechnique de Bucarest, II Année, no. 3, pp. 164–168.

[5] M Favard, C. R. Acad. Sci. Paris vol. 192 (1931) p. 716.

[6] T. Anghelutza, Mathematica, Cluj vol. 6 (1932) p. 140.

[7] H. L. Krall, *On the mean value theorem*, Amer. Math. Monthly vol. 42 (1935) pp. 604–606.

[8] N. Cioranescu, *Quelques propriétés* · · ·, Mathematica, Cluj vol. 9 (1935) pp. 184–193.

the contraction factor is the highest of the values x_i, which are solutions of the system[9] of $2m+2$ equations with $2m+2$ unknowns x_i, k_i:

$$\sum_{i=1}^{m+1} k_i x_i^{2q} = \frac{\nu}{\nu+2q}, \qquad x_{m+1} = 0, \qquad q = 0, 1, 2, \cdots, 2m. \tag{5'}$$

Here below we shall prove that the conditions (a) and (b) are always satisfied, we shall find the values of the contraction factors connected with the zeros of a sequence of Jacobi (more generally of Tchebycheff) polynomials, and shall see that our results are a generalization of Tchakaloff's and Biernacki's theorems.

Let[10] us put $x_i^2 = u_i$ $(i = 1, 2, \cdots, m)$. We observe that the second member of (5) can be written:

$$\frac{\nu}{\nu+2q} = \int_0^1 \nu u^{\nu+2q-1} du = \int_0^1 (\nu/2) u^{(\nu/2)-1} u^q du. \tag{6}$$

These are for $q = 0, 1, 2, \cdots, 2m-1$ the first m moments of the function

$$\psi(x) = (1/2) \int_0^x \nu u^{(\nu/2)-1} du;$$

therefore, we can write (5) as:

$$\sum_{i=1}^{m} k_i u_i^q = \int_0^1 u^q d\psi(u), \qquad q = 0, 1, 2, \cdots, 2m-1. \tag{7}$$

We observe that $\psi(x)$ is monotonically increasing in (0, 1); to such $\psi(x)$ correspond, as is well known,[11] a sequence $\{\phi_m(\nu; x)\}$ of orthonormal[12] polynomials, which, in turn, give rise to a mechanical quadrature formula:[11]

$$\int_0^1 G_{2m-1}(u) d\psi(u) = \sum_{i=1}^{m} H_i G_{2m-1}(l_i) \tag{8}$$

where $G_{2m-1}(x)$ represents an arbitrary polynomial of degree at most $2m-1$, the abscissas l_i are the zeros of $\phi_m(\nu; x)$, all real, distinct and

[9] This system is given by Cioranescu in his above mentioned paper for $m \leqq 2$.

[10] I am indebted for this interesting method to Professor J. A. Shohat, whose demonstration I follow very closely.

[11] J. A. Shohat, *Théorie générale des polynômes orthogonaux de Tchebycheff*, Mémorial des Sciences Mathématique, fascicule 66, pp. 8, 15.

[12] In this case they are Jacobi polynomials and coincide, for $\nu = 2$, with Legendre's polynomials.

inside (0, 1), and the coefficients H_i are all positive. It follows, for $G_{2m-1}(u)=u^q$,

$$\sum_{i=1}^{m} H_i l_i^q = \int_0^1 u^q d\psi(u), \qquad q = 0, 1, 2, \cdots, 2m-1. \tag{9}$$

Comparing (7) with (9) we see that a solution of (7) is given by

$$u_i = l_i, \qquad k_i = H_i, \tag{10}$$

with $0<l_i<1$, $H_i>0$, and, returning to (5):

$$x_i = \pm (l_i)^{1/2}, \qquad k_i = H_i, \qquad i = 1, 2, 3, \cdots, m. \tag{11}$$

Similar considerations on the m equations 2 to $m+1$ of (5′) (putting for instance $k_i x_i^2 = k_i'$) led also to m values $x_i^2 = l_i'$ $(i=1, 2, \cdots, m)$ to which we have only to add $x_{m+1}=0$, in order to have the complete set.

We may remark that if in $\Delta^{(n)}U=0$, $n=2m$, the moments of $\psi(x)$ are $\alpha_p=\nu/(\nu+2p)$ and the corresponding Jacobi polynomials form a complete sequence; if $n=2m-1$, the moments are $\alpha_p=\nu/(\nu+2(p+1))$ and the corresponding Jacobi polynomials form another complete sequence.

It follows that the systems (5) and (5′) have solutions, where all x_i are real and in absolute value less than one, and all k_i are positive; therefore, the conditions of Cioranescu are always satisfied.[13]

We could find the x_i also solving (5) by the method of Sylvester,[14] which, however, requires lengthy considerations of determinants. Its result is that the solutions x_i of (5), are the zeros of the polynomial:

$$E_{2m}(\nu; x) = \begin{vmatrix} \dfrac{x^2}{\nu} - \dfrac{1}{\nu+2} & \dfrac{x^2}{\nu+2} - \dfrac{1}{\nu+4} & \cdots & \dfrac{x^2}{\nu+2(m-1)} - \dfrac{1}{\nu+2m} \\ \dfrac{x^2}{\nu+2} - \dfrac{1}{\nu+4} & \dfrac{x^2}{\nu+4} - \dfrac{1}{\nu+6} & \cdots & \dfrac{x^2}{\nu+2m} - \dfrac{1}{\nu+2(m+1)} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{x^2}{\nu+2(m-1)} - \dfrac{1}{\nu+2m} & \dfrac{x^2}{\nu+2m} - \dfrac{1}{\nu+2(m+1)} & \cdots & \dfrac{x^2}{\nu+2(2m-2)} - \dfrac{1}{\nu+2(2m-1)} \end{vmatrix}. \tag{12}$$

[13] The present method has many points in common with that of J. A. Shohat, *On a certain formula of mechanical quadratures with non-equidistant ordinates*, Trans Amer. Math. Soc. vol. 31 (1929) pp. 449–450.

[14] For Sylvester's method see for instance T. Muir, *Theory of determinants*, vol. II, pp. 332–335. It may be noted that the formula (in Sylvester's notation): $a_{n+1}-a_n\sum\lambda_1+a_{n-1}\sum\lambda_1\lambda_2-\cdots=0$, which we meet there, generalizes an analogous equation indicated by P. Montel in his quoted paper.

Similarly, the solutions of (5′) are the zeros of

$$(12') \qquad E_{2m+1}(\nu; x) = x \begin{vmatrix} \frac{x^2}{\nu+2} & \frac{1}{\nu+4} & \frac{x^2}{\nu+4} & \frac{1}{\nu+6} & \cdots & \frac{x^2}{\nu+2m} & \frac{1}{\nu+2(m+1)} \\ \frac{x^2}{\nu+4} & \frac{1}{\nu+6} & \frac{x^2}{\nu+6} & \frac{1}{\nu+8} & \cdots & \frac{x^2}{\nu+2(m+1)} & \frac{1}{\nu+2(m+2)} \\ \cdot & & \cdot & & \cdots & \cdot & \\ \frac{x^2}{\nu+2m} & \frac{1}{\nu+2(m+1)} & \frac{x^2}{\nu+2(m+1)} & \frac{1}{\nu+2(m+2)} & \cdots & \frac{x^2}{\nu+2(2m-1)} & \frac{1}{\nu+4m} \end{vmatrix}.$$

These formulae have already been found for $\nu=1$ by Biernacki.[15]

We can thus make the general statement:

If the analytical function $U(P)$ of ν independent variables $y_1, y_2, \cdots, y_\nu$ satisfies the conditions $\Delta^{(n)}U=0$ and $\int_{\Sigma_\nu} U d\tau = 0$, there is inside the hypersphere of radius $x_M R$ at least one point P_0 which makes $U(P_0)=0$, R being the radius of Σ_ν and $x_M<1$ the square root of the highest zero of the Jacobi polynomial[16] *$\phi_m(\nu; x)$, where $m=n/2$ or $m=(n-1)/2$, according to the parity of n; the same value x_M is also the highest zero of the polynomial*[17] *$E_n(\nu; x)$.*

Taking into account the formulae of Tchebycheff's polynomials:[18]

$$(13) \qquad \Phi_n(x) = \frac{1}{\Delta_n(\psi)} \begin{vmatrix} \alpha_0 & \alpha_1 \cdots \alpha_n \\ \alpha_1 & \alpha_2 \cdots \alpha_{n+1} \\ \cdot & \cdot \cdots \cdot \\ \alpha_{n-1} & \alpha_n \cdots \alpha_{2n-1} \\ 1 & x \cdots x^n \end{vmatrix} = x^n - \delta_{n,n-1}x^{n-1} + \delta_{n,n-2}x^{n-2} - \cdots$$

with

$$(14) \qquad \Delta_n(\psi) = \begin{vmatrix} \alpha_0 & \alpha_1 \cdots \alpha_{n-1} \\ \alpha_1 & \alpha_2 \cdots \alpha_n \\ \cdot & \cdot \cdots \cdot \\ \alpha_{n-1} & \alpha_n \cdots \alpha_{2n-2} \end{vmatrix}$$

and the formulae of Szegö:[19]

[15] M. Biernacki, loc. cit. p. 166.

[16] We can write immediately $\phi_m(\nu; x)$, knowing the moments of $\psi(x)$.

[17] As we shall see, the polynomials $E_n(\nu; x)$ form a *single* sequence of Tchebycheff polynomials, without distinction as to whether n is even or odd.

[18] J. A. Shohat, loc. cit. pp. 3–5.

[19] G. Szegö, *Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören*, Math. Zeit. vol. 9 (1921) p. 218.

$$(15)\qquad \Delta_{2m}(\psi)\Phi_{2m}(x)=\begin{vmatrix}\alpha_0 & 0 & \cdots & \alpha_{2m}\\ 0 & \alpha_2 & \cdots & 0\\ \alpha_2 & 0 & \cdots & \alpha_{2m+2}\\ \cdot & \cdot & \cdots & \cdot\\ 0 & \alpha_{2m} & \cdots & 0\\ 1 & x & \cdots & x^{2m}\end{vmatrix}=\begin{vmatrix}+\alpha_0x & -\alpha_2 & \cdots & -\alpha_{2m}\\ -\alpha_2 & +\alpha_3x & \cdots & +\alpha_{2m}x\\ -\alpha_3x & +\alpha_4 & \cdots & +\alpha_{2m+2}\\ \cdot & \cdot & \cdots & \cdot\\ +\alpha_{2m-3}x & -\alpha_{2m} & \cdots & -\alpha_{4m-2}\\ -\alpha_{2m} & +\alpha_{2m}x & \cdots & +\alpha_{4m-3}x\end{vmatrix},$$

$$(15')\qquad \Delta_{2m+1}(\psi)\Phi_{2m+1}(x)=\begin{vmatrix}\alpha_0 & 0 & \cdots & 0\\ 0 & \alpha_2 & \cdots & \alpha_{2m+2}\\ \alpha_2 & 0 & \cdots & 0\\ \cdot & \cdot & \cdots & \cdot\\ \alpha_{2m} & 0 & \cdots & 0\\ 1 & x & \cdots & x^{2m+1}\end{vmatrix}=\begin{vmatrix}+\alpha_0x & -\alpha_2 & \cdots & +\alpha_{2m}x\\ -\alpha_2 & +\alpha_3x & \cdots & -\alpha_{2m+2}\\ +\alpha_3x & -\alpha_4 & \cdots & +\alpha_{2m+3}x\\ \cdot & \cdot & \cdots & \cdot\\ -\alpha_{2m} & +\alpha_{2m}x & \cdots & -\alpha_{4m}\\ +\alpha_{2m}x & -\alpha_{2m+3} & \cdots & +\alpha_{4m}x\end{vmatrix},$$

we find easily:

$$(16)\qquad \Delta_{2m}(\psi)\Phi_{2m}(x)=\begin{vmatrix}\alpha_0 & 0 & \cdots & \alpha_{2m}\\ 0 & \alpha_2 & \cdots & 0\\ \alpha_2 & 0 & \cdots & \alpha_{2m+2}\\ \cdot & \cdot & \cdots & \cdot\\ 0 & \alpha_{2m} & \cdots & 0\\ 1 & x & \cdots & x^{2m}\end{vmatrix}=C_{2m+1}\begin{vmatrix}\alpha_0x^2-\alpha_2 & \alpha_3x^2-\alpha_4 & \cdots & \alpha_{2m-3}x^2-\alpha_{2m}\\ \alpha_3x^2-\alpha_4 & \alpha_4x^2-\alpha_6 & \cdots & \alpha_{2m}x^2-\alpha_{2m+2}\\ \cdot & \cdot & \cdots & \cdot\\ \alpha_{2m-3}x^2-\alpha_{2m} & \alpha_{2m}x^2-\alpha_{2m+2} & \cdots & \alpha_{4m-4}x^2-\alpha_{4m-2}\end{vmatrix},$$

$$(16')\qquad \Delta_{2m+1}(\psi)\Phi_{2m+1}(x)=\begin{vmatrix}\alpha_0 & 0 & \cdots & 0\\ 0 & \alpha_2 & \cdots & \alpha_{2m+2}\\ \alpha_2 & 0 & \cdots & 0\\ \cdot & \cdot & \cdots & \cdot\\ \alpha_{2m} & 0 & \cdots & 0\\ 1 & x & \cdots & x^{2m+1}\end{vmatrix}=C_{2m+2}x\begin{vmatrix}\alpha_2x^2-\alpha_4 & \alpha_4x^2-\alpha_6 & \cdots & \alpha_{2m}x^2-\alpha_{2m+2}\\ \alpha_4x^2-\alpha_6 & \alpha_6x^2-\alpha_8 & \cdots & \alpha_{2m+2}x^2-\alpha_{2m+4}\\ \cdot & \cdot & \cdots & \cdot\\ \alpha_{2m}x^2-\alpha_{2m+2} & \alpha_{2m+2}x^2-\alpha_{2m+4} & \cdots & \alpha_{4m-2}x^2-\alpha_{4m}\end{vmatrix}$$

where

$$(17)\qquad C_{2m}=\begin{vmatrix}\alpha_0 & \alpha_2 & \cdots & \alpha_{2m-2}\\ \alpha_2 & \alpha_4 & \cdots & \alpha_{2m}\\ \cdot & \cdot & \cdots & \cdot\\ \alpha_{2m-2} & \alpha_{2m} & \cdots & \alpha_{4m-4}\end{vmatrix},$$

$$(17) \qquad C_{2m+1} = \begin{vmatrix} \alpha_2 & \alpha_4 & \cdots & \alpha_{2m} \\ \alpha_4 & \alpha_6 & \cdots & \alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m} & \alpha_{2m+2} & \cdots & \alpha_{4m-2} \end{vmatrix}.$$

Equating in (16) the coefficients of x^{2m}, and in (16′) the coefficients of x^{2m+1}, we obtain,[20] taking into account (13):

$$\Delta_{2m}(\psi) = C_{2m}C_{2m+1}$$

and

$$\Delta_{2m+1}(\psi) = C_{2m+1}C_{2m+2}$$

or generally

$$(18) \qquad \Delta_n(\psi) = C_nC_{n+1}.$$

Let us put now:

$$(19) \qquad \alpha_{2m} = \frac{1}{\nu + 2m}, \qquad \alpha_{2m+1} = 0.$$

Then the C_n, easily calculable, being all positive, (18) shows that $\Delta_n(\psi) > 0$. Therefore, the conditions of Hamburger[21] being satisfied, there is a monotonically increasing function $\Psi(x)$[22], whose moments are (19). Comparing (12), (12′) with (16), (16′), we see that the polynomials $E_n(\nu; x)$ are just the sequence of polynomials of Tchebycheff corresponding to $\Psi(x)$. As they form a sequence of Sturm[23] and as $E_n(\nu; 1) > 0$, $E_{2m}(\nu; 0) = (-1)^m C_{2m+1}$, $\lim_{x=0} (1/x)E_{2m+1}(\nu; x) = (-1)^m D_{2m}$, with D_{2m} an easily calculable positive determinant, we find also in this way that all the x_i, zeros of $E_n(\nu; x)$ are real, distinct and inside $(-1, +1)$.[24]

The function U, polyharmonical of order n, can be in particular a polynomial of degree $2n-1$, of ν independent variables. If a polynomial is of an even degree, it may be considered for our purpose as

[20] I have learned that the same formulae have been derived by C. Rees in his thesis on *Elliptic orthogonal polynomials*.

[21] H. Hamburger, *Über eine Erweiterung des Stieltjeschen Momentenproblems*, Math. Ann. vol. 81 (1920) pp. 235–319; vol. 82 (1921) pp. 120–164 and pp. 168–187.

[22] Taking into account the further conditions of the problem, we easily find this function, which is: $d\Psi(x) = (1/2)|x|^{\nu-1}dx$ for $|x| \leqq 1$, $d\Psi(x) = 0$ for $|x| > 1$.

[23] See also Brioschi, *Théorie des déterminants*, pp. 85–86.

[24] By solving completely the system (5) it can be proved also that $k_1 > 0$, but this requires a very long calculation, whereas the ingenious method of J. A. Shohat yields immediately the result. Some rather complicated expressions of the k_i in function of ν are given, for small values of n, in the quoted paper of N. Cioranescu in Mathematica, Cluj vol. 9 (1935) pp. 191–192.

of the next higher odd degree, with its first coefficient equal to zero.[25]

For $\nu=1$ we fall back on the known theorems of P. Montel, Tchakaloff[26] and Biernacki.[27] In fact, the formulae (12), (12′), with (16), (16′) and (19) show us [28] that for $\nu=1$, $E_n(1;x)$ are but the polynomials $P_n(x)$ of Legendre; moreover:

$$(3'') \qquad \Delta^{(n)}U(y)=\frac{d^{2n}U(y)}{dy^{2n}}=0$$

if the degree of U is at most $2n-1$. Applying our results from above to this case we find that if x_M is the highest zero of Legendre's polynomial $P_n(x)$ and

$$(4'') \qquad \int_a^b U(y)dy=0$$

where $U(y)$ is a polynomial of degree at most $2n-1$, then $U(y)$ has at least one zero inside the interval $(1/2)(b+a)\pm x_M(b-a)/2$. These are precisely the results of Tchakaloff and Biernacki and our main statement from p. 545 can be considered as a generalization of the precisions which the above mentioned authors have brought to the mean value theorem.

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[25] This has been shown for $\nu=1$ by P. Montel in his quoted paper. A slight difference can be noted between his statement and ours, because P. Montel's refers not to the function $U(y)$, but to $F(y)=\int_{y_0}^{y}U(y)dy$.

[26] Tchakaloff, loc. cit. p. 34.

[27] Biernacki, loc. cit. p. 167–168.

[28] We may remark that the formulae (13), (13′), (16) and (16′) attest the equivalence of the solutions found by the two authors in the problem of the contraction factor, in spite of their different form.

ALTERNATIVE ALGEBRAS OVER AN ARBITRARY FIELD

R. D. SCHAFER[1]

The results of M. Zorn concerning alternative algebras[2] are incomplete over modular fields since, in his study of alternative division algebras, Zorn restricted the characteristic of the base field to be not two or three. In this paper we present first a unified treatment of alternative division algebras which, together with Zorn's results, permits us to state that any alternative, but not associative, algebra A over an arbitrary field F is central simple (that is, simple for all scalar extensions) if and only if A is a Cayley-Dickson algebra[3] over F.

A. A. Albert in a recent paper, *Non-associative algebras* I: *Fundamental concepts and isotopy*,[4] introduced the concept of isotopy for the study of non-associative algebras. We present in the concluding section of this paper theorems concerning isotopes (with unity quantities) of alternative algebras. The reader is referred to Albert's paper, moreover, for definitions and explanations of notations which appear there and which, in the interests of brevity, have been omitted from this paper.

1. **Alternative algebras.** A distributive algebra A is called an *alternative algebra* if $ax^2=(ax)x$ and $x^2a=x(xa)$ for all elements a, x in A. That is, in terms of the so-called right and left multiplications, A is alternative if $R_{x^2}=(R_x)^2$ and $L_{x^2}=(L_x)^2$.

The following lemma, due to R. Moufang,[5] and the Theorem of Artin are well known.

LEMMA 1. *The relations* $L_xR_xR_y=R_{xy}L_x$, $R_xL_xL_y=L_{yx}R_x$, *and* $R_xL_{xy}=L_yL_xR_x$ *hold for all* a, x, y *in an alternative algebra* A.

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[1] This paper is the essential portion of the author's doctoral dissertation, written at The University of Chicago under the direction of Professor A. A. Albert, whose kindness the author gratefully acknowledges.

[2] M. Zorn, *Theorie der Alternativen Ringe*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930) pp. 123–147; *Alternativkörper und Quadratische Systeme*, ibid. vol. 9 (1933) pp. 395–402. See M. Zorn, *Alternative rings and related questions* I: *Existence of the radical*, Ann. of Math. (2) vol. 42 (1941) pp. 676–686, to verify that the Peirce decomposition in Zorn's first paper is valid for F of characteristic two.

[3] Defined over an arbitrary field by A. A. Albert in his *Quadratic forms permitting composition*, Ann. of Math. (2) vol. 43 (1942) pp. 161–177.

[4] Ann. of Math. (2) vol. 43 (1942) pp. 685–708.

[5] R. Moufang, *Zur Struktur von Alternativkörpern*, Math. Ann. vol. 110 (1934) pp. 416–430.

THEOREM OF ARTIN. *The subalgebra generated by any two elements of an alternative algebra A is associative.*

It follows from the Theorem of Artin that if x is any element of an alternative algebra A over F, and if $f(\lambda)$ is any polynomial in λ with coefficients in F, then $f(R_x)=R_{f(x)}$ and $f(L_x)=L_{f(x)}$.

Moreover, if an alternative algebra A contains a right nonsingular element x and a left nonsingular element y, then A has a unity quantity. For the identity transformation I is then a polynomial in R_x with coefficients in F, and the correspondence $x\rightarrow R_x$ is one-to-one. Thus the unity quantity of A and the inverse x^{-1} exist, and are polynomials in x, and $R_{x^{-1}}=(R_x)^{-1}$. Clearly these results hold for an alternative division algebra A.

In this paper we shall require the following lemma.

LEMMA 2. *In an alternative division algebra A, the norm of a product xy is equal to the product of the norms of x and y.*

For if $x=0$ the lemma is obvious. Otherwise L_x is nonsingular, and $L_xR_xR_y=R_{xy}L_x$ by Lemma 1. Thus $|L_x|\cdot|R_x|\cdot|R_y|=|R_{xy}|\cdot|L_x|$, and $|R_x|\cdot|R_y|=|R_{xy}|$. The conclusion follows.

Since any simple algebra A over F is central simple over its transformation center, the determination of all simple alternative algebras consists of a determination of those which are central simple. Zorn's results imply that a central simple alternative algebra over an arbitrary field is either (1) a division algebra, (2) an associative algebra, or (3) a Cayley-Dickson algebra with divisors of zero.[6] We are led directly to this theorem.

THEOREM 1. *Let A be an alternative, but not associative, central division algebra over F. Then A is an algebra of degree two and order eight over F.*

For there exists a scalar extension A_K of A such that A_K over K is not a division algebra. Since A_K is not associative, A_K is a Cayley-Dickson algebra (with divisors of zero) over K. Hence A is of degree two and order eight over F.

2. **Alternative division algebras of degree two.** We are able to make a study of alternative division algebras of degree two which is independent of Zorn's results, and (although a portion of the result is indicated in Theorem 1) we shall do so. For the proof of Theorem 2 we require the two lemmas which follow.

[6] Zorn's so-called "vector matrix algebra" is (over any field, including those of characteristic two) merely a Cayley-Dickson algebra with divisors of zero.

LEMMA 3. *Let the principal function of any element x of an alternative division algebra of degree two over F be $x^2-t(x)\cdot x+n(x)$, where $t(x)$ and $n(x)$ are in F. Then the linear transformation*

$$S: \quad x \leftrightarrow xS = t(x) - x \tag{1}$$

is an involution of A such that $x+xS=t(x)$, and $x(xS)=(xS)x=n(x)$.

The conclusions are trivial except for showing that S is an involution of A. Clearly $S^2=I$. By Lemma 2 we have $(xy)\cdot(xy)S=n(xy)=n(x)\cdot n(y)=(x\cdot xS)\cdot n(y)$. It follows from the Theorem of Artin that $(xy)S=(xy)^{-1}\cdot(xy)\cdot(xy)S=y^{-1}x^{-1}x(xS)\cdot n(y)=n(y)\cdot y^{-1}(xS)=(yS\cdot y)y^{-1}(xS)=yS\cdot xS$. Hence S is an involution of A.

LEMMA 4. *Let B be an alternative algebra of order $2s$ over F defined as the supplementary sum $B=f_1D+\cdots+f_sD$, where $D=(1, u_2)$, $u_2^2=u_2+\alpha$, $-4\alpha\neq 1$, $f_1=1$, $f_i^2=\gamma_i\neq 0$, γ_i in F, $d\cdot f_i=f_i\cdot dS$ for all d in D, S as in (1), $(i=2,\cdots,s)$. Then, if B is a proper subalgebra of an alternative division algebra A of degree two over F, there exists an element g in A, but not in B, such that*

$$g^2=\gamma\neq 0,\ \gamma \text{ in } F, \text{ and } xg=g(xS) \tag{2}$$

holds for all x in B.

It is evident that $f_iD\cdot f_jD\subseteq D$ and that the intersection $(f_iD\cdot f_kD)\cap D=0$ for $j\neq k$, $j, k=1, 2,\cdots, s$. In order to establish (2) it is sufficient to prove the existence of g with the trace $t(xg)=0$ for all x of B. Now if y is any element of B, then $y=d_1+f_2d_2+\cdots+f_sd_s$ with d_i in D. But $t(f_jd_i)=0$ for $j\neq 1$. Hence $t(y)=t(d_1)$.

Let v be an element of A not in B, and write $g=(\lambda_1+\lambda_2u_2)+f_2(\lambda_3+\lambda_4u_2)+\cdots+f_s(\lambda_{2s-1}+\lambda_{2s}u_2)+v$, where the λ_i are undetermined coefficients in F. Denote $t(f_jv)$ by μ_{2j-1} and $t(f_ju_2\cdot v)$ by μ_{2j}. Then the existence of g satisfying (2) is equivalent to the existence of λ_i $(i=1,\cdots,2s)$ in F such that

$$\begin{aligned}
t(g) &= 2\lambda_1+\lambda_2+\mu_1=0,\\
t(u_2g) &= \lambda_1+(1+2\alpha)\lambda_2+\mu_2=0,\\
t(f_jg) &= 2\gamma_j\lambda_{2j-1}+\gamma_j\lambda_{2j}+\mu_{2j-1}=0, \quad j=2,\cdots,s,\\
t(f_ju_2\cdot g) &= \gamma_j\lambda_{2j-1}-2\alpha\gamma_j\lambda_{2j}+\mu_{2j}=0, \quad j=2,\cdots,s.
\end{aligned}$$

But the determinant Δ of the coefficients of this system of linear equations is $\Delta=(1+4\alpha)\cdot\prod_2^s\gamma_j^2(-1-4\alpha)\neq 0$. Hence the desired solutions λ_i exist.

An algebra Q is called a *quaternion algebra* if $Q=(1, u_2, u_3, u_4)$, $u_4=u_3u_2$, $u_2^2=u_2+\alpha$, $u_3^2=\beta$, $u_2u_3=u_3(1-u_2)$, where α and $\beta\neq 0$ are in F,

$-4\alpha \neq 1$. Moreover, Q is a division algebra if and only if there exist no λ and μ in F such that $\beta = \lambda^2 + \lambda\mu - \alpha\mu^2$.

An algebra C is called a *Cayley-Dickson algebra* if $C = Q + gQ$, with elements $z = x + gy$, where x, y are quaternions, and multiplication is defined by

$$(3)\quad (x_1 + gy_1)(x_2 + gy_2) = (x_1x_2 + \gamma y_2 \cdot y_1S) + g(x_1S \cdot y_2 + x_2y_1),$$

where $g^2 = \gamma \neq 0$ in F and S is the involution (1) of Q.

THEOREM 2. *An alternative algebra A over F is a division algebra of degree two over F if and only if A is one of the following:*

(a) *a separable quadratic field, or an inseparable field of exponent two,*

(b) *a quaternion division algebra Q, or*

(c) *a Cayley-Dickson algebra $C = Q + gQ$, where Q is a division algebra and there exist no $\lambda, \mu, \rho, \sigma$ in F such that*

$$(4)\qquad \gamma = \lambda^2 + \lambda\mu - \alpha\mu^2 - \beta\rho^2 - \beta\rho\sigma + \alpha\beta\sigma^2.$$

If A is generated by less than three elements, then A is associative and is either (a) or (b). Otherwise, if the characteristic of F is not two, A contains a quaternion subalgebra Q as in (b), and an element v which is not in Q. If the characteristic of F is two, consider two cases: if A is commutative, then A is associative[7] and is either (a) or (b). If A is not commutative, there exist two noncommutative elements x, y. These generate an associative, noncommutative subalgebra (a quaternion algebra Q) of A. Also there exists an element v of A which is not in Q.

The algebra Q is a particular example, $s = 2$, of the algebra B defined in Lemma 4, $f_2 = u_3$, $\gamma_2 = \beta$. Thus there exists an element g in A, but not in $Q = B$, satisfying (2). Then A contains $A_0 = B + gB$, the elements z of A_0 being expressible uniquely as $z = x + gy$ for x, y in B. We make effective use of equation (2), Lemma 1, and the Theorem of Artin in proving that A_0 is an algebra in which multiplication is defined by (3).

For $(gy_1)(gy_2) = (gy_1)(y_2S \cdot g) = [g(y_1 \cdot y_2S)]g = [(y_2 \cdot y_1S)g]g = (y_2 \cdot y_1S)g^2 = \gamma y_2 \cdot y_1S$. Also from the fact that $y_1S = \gamma^{-1}y_1S\gamma = \gamma^{-1}y_1S \cdot g^2 = \gamma^{-1}(y_1S \cdot g)g = \gamma^{-1}(gy_1)g$, it follows that $g(x_2y_1) = (y_1S \cdot x_2S)g = [\{\gamma^{-1}(gy_1)g\}x_2S]g = \gamma^{-1}(gy_1)[g(x_2S \cdot g)] = \gamma^{-1}(gy_1)[g(gx_2)] = \gamma^{-1}(gy_1) \cdot (\gamma x_2) = (gy_1)x_2$. But then $x_1 = \gamma^{-1}(g \cdot x_1S)g$ and $g(y_2S \cdot x_1) = (gx_1) \cdot y_2S$ also. Hence $x_1(gy_2) = [\gamma^{-1}(g \cdot x_1S)g](gy_2) = \gamma^{-1}[g(x_1S \cdot g)](y_2S \cdot g) = \gamma^{-1}[g\{(x_1S \cdot g)y_2S\}]g = \gamma^{-1}[g\{(gx_1)y_2S\}]g = \gamma^{-1}[g\{g(y_2S \cdot x_1)\}]g = \gamma^{-1}(\gamma y_2S \cdot x_1)g = g(x_1S \cdot y_2)$. Hence equation (3) holds.

[7] A commutative alternative algebra over a field of characteristic not three is associative. See M. Zorn, *Theorie der Alternativen Ringe*, op. cit.

Having shown that any alternative division algebra of degree two, which is not associative, must contain the algebra A_0, we need now only to prove that A_0 is actually an alternative algebra and is not a proper subalgebra of any alternative division algebra of degree two. We may readily verify these conclusions if we refer to the matrices corresponding to the linear transformations we have used. Thus, for $z=x+gy$ in A_0,

$$(5)\qquad R_z=\begin{pmatrix} R_x & SR_y \\ \gamma SL_y & L_x \end{pmatrix},\qquad L_z=\begin{pmatrix} L_x & R_y \\ \gamma R_{yS} & L_{Sx} \end{pmatrix}$$

where the matrices $R_x, L_x, \cdots, S$ are the 4-by-4 matrices of the linear transformations induced on Q by the corresponding transformations on A_0. Computation, involving the associativity of Q, yields $R_{z^2}=(R_z)^2$ and $L_{z^2}=(L_z)^2$. Hence the Cayley-Dickson algebra $C=A_0$ is alternative. If C were a proper subalgebra of A, then A would contain a new $A_0=C+hC$ with multiplication defined by (5), x and y being in C. But computation with the matrices above reveals that if A_0 were alternative, C would be associative, which it is not.

Condition (4) follows from Lemmas 2 and 3. Let $z=x+gy$ with x, y in Q, and let S be defined by (1). Then $n(z)=z\cdot zS=(x+gy)(xS+yS\cdot gS)$ $=(x+gy)(xS-gy)=x\cdot xS-\gamma y\cdot yS=n(x)-\gamma n(y)$. Now C is a division algebra if and only if $n(z)\neq 0$ for every nonzero z in C. But $n(x)-\gamma n(y)=0$ if and only if $\gamma=n(x)\,[n(y)\,]^{-1}=n(xy^{-1})=n(v)$, the norm of v for some v in Q. Let $v=\lambda+\mu u_2+\rho u_3+\sigma u_4$. Then C is a division algebra if and only if there exist no $\lambda, \mu, \rho, \sigma$ in F such that $\gamma=n(v)=v\cdot vS=\lambda^2+\lambda\mu-\alpha\mu^2-\beta\rho^2-\beta\rho\sigma+\alpha\beta\sigma^2$.

We may combine the results of Zorn and Theorems 1 and 2 in the following manner and say: an alternative, but not associative, algebra A over an arbitrary field F is central simple if and only if A is a Cayley-Dickson algebra over F.

3. **Isotopes of alternative algebras.** Albert has proved that an algebra A with a unity quantity is associative if and only if every isotope of A with a unity quantity is associative and is equivalent to A. We consider here the corresponding problem for alternative algebras.

THEOREM 3. *Let A be an alternative algebra, and B be an isotope of A with a unity quantity. Then A has a unity quantity, and B is alternative.*

For B is equivalent to a principal isotope A_0 of A, in which products (a, x) are defined by $(a, x) = aR_x^{(0)}$ with $R_x^{(0)} = PR_{xQ}$ for nonsingular transformations P, Q. (In terms of left multiplications, $(x, a) = aL_x^{(0)}$ with $L_x^{(0)} = QL_{xP}$.) Let e be the unity quantity of A_0 and $h = eQ$. Then $I = R_e^{(0)} = PR_h$, and R_h is nonsingular. Similarly if $k = eP$, then L_k is nonsingular, and it follows from the proof in §1 that the algebra A has a unity quantity.

Now $P = R_h^{-1} = R_{h^{-1}}$. Hence $R_x^{(0)} = R_{h^{-1}}R_{xQ} = L_h R_{h^{-1}(xQ)}L_{h^{-1}}$ by Lemma 1. That is, $R_x^{(0)} = HR_{(xT)H}H^{-1}$ where $H = L_h$ and $T = QL_h^{-2}$. Hence A_0 is equivalent to an isotope A_1 of A in which products are denoted by $[a, x] = aR_x^{(1)}$ where $R_x^{(1)} = R_{xT}$. Also $[x, a] = aL_x^{(1)}$ where $L_x^{(1)} = TL_x$. Let f be the unity quantity of A_1. Then $I = L_f^{(1)} = TL_f$ and $T = L_f^{-1} = L_{f^{-1}}$. Hence $R_x^{(1)} = R_{f^{-1}x}$. Therefore $R_{[x,x]}^{(1)} = R_{xR_x^{(1)}}^{(1)} = R_{xR_{f^{-1}x}}$ $= R_{f^{-1}\{x(f^{-1}x)\}} = R_{(f^{-1}x)^2} = R_{f^{-1}x}R_{f^{-1}x} = R_x^{(1)}R_x^{(1)}$ since A is alternative. Since B is equivalent to A_1, it follows that half of the alternative law holds in B.

But similarly B is equivalent to an isotope A_2 of A in which products are denoted by $\{x, a\} = aL_x^{(2)}$ where $L_x^{(2)} = L_{xc^{-1}}$, the element c being the unity quantity of A_2. Then $L_{[x,x]}^{(2)} = L_x^{(2)}L_x^{(2)}$, and the second half of the alternative law holds in B.

We complete the study of isotopy for simple alternative algebras by proving the following theorem.

THEOREM 4. *Let A be a Cayley-Dickson algebra, and B be any isotope of A with a unity quantity. Then B is equivalent to A.*

Any isotope with a unity quantity of a central simple algebra A with a unity quantity is also central simple. Therefore, any isotope B with a unity quantity of a Cayley-Dickson algebra A is also a Cayley-Dickson algebra. We shall show that B is equivalent to A.

For B is equivalent to an algebra A_1 in which products are denoted by $[a, x] = aR_x^{(1)}$ where $R_x^{(1)} = R_{f^{-1}x}$, the element f being the unity quantity of A_1. Now f, as an element in A, is contained in some quaternion subalgebra Q of A. Let x range over Q, and R be the subspace of A_1 consisting of all elements fx. Then $x \leftrightarrow fx$ is an equivalence of Q and R. For since Q contains f and is associative, $[fx, fy] = fxR_{f^{-1}fy}$ $= f(xy)$ for all x, y in Q.

Let S be the involution of Q defined by (1), and let $z = fx$. Then the transformation

$$U: \quad z \leftrightarrow zU = f(xS)$$

is the corresponding involution of R. Now $A = Q + gQ$ as in (3). Also $A_1 = R + wR$, where $w = fg$ (and where the multiplication defining wR

is of course the multiplication in A_1). By the proof of Theorem 2, in order to show the equivalence of A and A_1 it is sufficient to show that $[w, w]=\gamma f$ and $[w, z]=[zU, w]$ for every z of R. But $[w, w]=w(f^{-1}w)=(fg)(f^{-1}fg)=fg^2=\gamma f$, and $[w, z]=w(f^{-1}z)=(fg)(f^{-1}fx)=(fg)x=g(x\cdot fS)=(f\cdot xS)g=(f\cdot xS)(f^{-1}fg)=zU(f^{-1}w)=[zU, w]$. This proves the theorem.

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ON FIBRE SPACES. I

RALPH H. FOX

In subsequent papers I propose to investigate various properties of fibre spaces.[1] The object of the fundamental Hurewicz-Steenrod definition[1] is to state a minimum[2] set of readily verifiable conditions under which the covering homotopy theorem[1] holds. An apparent defect of their definition is that it is not topologically invariant. In fact, for topological space X and metrizable non-compact space B the property "X is a fibre space over B" depends on the metric of B. The object of this note is to give a topologically invariant definition of fibre space and to show that (when B is metrizable) X is a fibre space over B in this sense if and only if B has a metric in which X is a fibre space over B in the sense of Hurewicz-Steenrod. Since the definition of fibre space is controlled by the covering homotopy theorem, an essential part of my program is to give a topologically invariant definition of uniform homotopy.

Let π be a continuous mapping of a topological space X into another topological space B. Let $\Delta=\Delta(B)$ denote the diagonal set $\sum_{b\in B}(b, b)$ of the product space $B\times B$ and let $\bar{\pi}$ denote the mapping of $X\times B$ into $B\times B$ which is induced by the mapping π according to the rule $\bar{\pi}(x, b)=(\pi(x), b)$. Thus the graph G of π is the set $\bar{\pi}^{-1}(\Delta)$, and $\bar{\pi}^{-1}(U)$ is a neighborhood of G whenever U is a neighborhood of Δ.

Any neighborhood U of Δ determines uniquely a covering of B by neighborhoods $N_U(b)$ according to the rule $b'\in N_U(b)$ when $(b, b')\in U$.

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[1] W. Hurewicz and N. E. Steenrod, Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) p. 61.

[2] How well they succeeded in this will be indicated in my next communication.

However not every covering of B by neighborhoods need arise in this fashion—although the star neighborhoods of any open covering of B may always be so generated.

A *slicing function* ϕ for π is any continuous mapping defined over $\bar{\pi}^{-1}(U)$ for some neighborhood U of Δ, with values in X, which satisfies the conditions

$$\pi\phi(x, b) = b,$$
$$\phi(x, \pi(x)) = x,$$

whenever ϕ is defined. I shall call π a *fibre mapping* relative to U if it has a slicing function defined over $\bar{\pi}^{-1}(U)$. If π is a fibre mapping I shall say that X is a *fibre space* over the subset $\pi(X)$ of B. Since U is a neighborhood of Δ, $\pi(X)$ is open and closed in B.

This new definition is equivalent to the old one if the base space is compact metric (so that the Hurewicz-Steenrod definition is topologically invariant in this case). In fact, for metric space B, let σ_ϵ denote that neighborhood of Δ which determines the covering of B by ϵ-spheres. Clearly X is a fibre space (relative to π) over the metric space $\pi(X)$ in the sense of Hurewicz-Steenrod if and only if π has a slicing function defined over $\bar{\pi}^{-1}(\sigma_\epsilon)$ for some $\epsilon>0$. Hence, *if π is a fibre mapping and $\pi(X)$ is compact metrizable then X is a fibre space over $\pi(X)$ in the sense of Hurewicz-Steenrod no matter how $\pi(X)$ is metrized.*

Now let B denote an arbitrary metrizable space, let U be a neighborhood of Δ and let π be a fibre mapping whose slicing function is defined over $\bar{\pi}^{-1}(U)$. For simplicity, assume also that $\pi(X)=B$. To show that X is a fibre space in the sense of Hurewicz-Steenrod when B is properly metrized it is clearly sufficient to so metrize B that $\sigma_\epsilon \subset U$ for some $\epsilon>0$.

LEMMA.[3] *If B is metrizable and U is an open neighborhood of $\Delta(B)$ then B can be so metrized that $\sigma_1 \subset U$.*

Choose any random metric d for B. Since $B\times B$ is metric, hence normal, it is possible to define a continuous function $f\in[0, 1]^{B\times B}$ such that

$$f(b, b_0) = \begin{cases} 0 & \text{when } (b, b_0) \in \Delta, \\ 1 & \text{when } (b, b_0) \in B \times B - U. \end{cases}$$

Let ϕ denote the (continuous) mapping $b\to f_b$, where $f_b(b_0)=f(b, b_0)$.

[3] This proof is modelled after a proof in André Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Actualités Scientifiques et Industrielles, no. 551, 1938, p. 15.

The graph $B' = \sum_{b \in B}(b, \phi(b))$ of ϕ is homeomorphic to B. The metric of B' is induced by the metric of the product $B \times [0, 1]^B$ and is given by the formula

$$\delta(b_1', b_2') = \{d^2(b_1, b_2) + d^2(\phi(b_1), \phi(b_2))\}^{1/2},$$

where b and b' denote corresponding points of B and B'. If now $(b_1', b_2') \in \sigma_1$ then $\delta(b_1', b_2') < 1$, hence $d(\phi(b_1), \phi(b_2)) < 1$, hence $\sup_{b \in B} |f(b_1, b) - f(b_2, b)| < 1$. It follows that $f(b_1, b_2) = |f(b_1, b_2) - f(b_2, b_2)| < 1$, so that $(b_1, b_2) \in U$ and $(b_1', b_2') \in U'$.

THEOREM. *If π is a fibre mapping and B is metrizable then the metric of B can be so chosen that X is a fibre space over $\pi(X)$ (relative to π) in the sense of Hurewicz and Steenrod.*

I conclude by defining uniform homotopy and stating the covering homotopy theorem for general fibre spaces. If h is a homotopy in B of a space Y and U is a neighborhood of Δ I shall say that h is *uniform* with respect to U if there is a $\delta > 0$ such that $|t - t'| < \delta$ implies that $\sum_{y \in Y}(h(y, t), h(y, t')) \subset U$. Let $E_\delta = \sum_{0 \leq t, t' \leq 1, |t-t'| < \delta} \sum_{y \in Y}(h(y, t), h(y, t'))$, so that $E_0 \subset \Delta$ and $E_1 \subset B \times B$. Clearly the neighborhoods U with respect to which h is uniform are those which contain an E_δ for some $\delta > 0$. Thus h is always uniform with respect to $B \times B$; in the event that Y is compact h is uniform with respect to every neighborhood U. I shall call a homotopy h^* in X a *covering homotopy* (with respect to π) if

(1) $\pi h^* = h$,

(2) $h^*_{[0,1]}(y)$ degenerates to a point whenever $h_{[0,1]}(y)$ degenerates to a point.

I shall refer to the mappings h_0 and h_0^* as the initial values of the homotopies h and h^*, respectively. With these notations the covering homotopy theorem for fibre mappings reads thus.

THEOREM. *Given a fibre mapping $\pi \in B^X$ relative to U, a mapping $g \in X^Y$ and a homotopy h in B, uniform with respect to U, with initial value πg, there exists a covering homotopy h^* in X with initial value g.*

The covering homotopy h^* is constructed stepwise[1] and is easily seen to be uniform with respect to $U^* = \bar{\pi}^{-1}(U)$ where $\bar{\pi}(x, x') = (\pi(x), \pi(x'))$. Of course if U is a σ_s the neighborhood U^* of $\Delta(X)$ need not be a $\sigma_s(X)$.

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CHAINS IN PARTIALLY ORDERED SETS

OYSTEIN ORE

1. **Introduction.** Dedekind [1] in his remarkable paper on *Dualgruppen* was the first to analyze the axiomatic basis for the theorem of Jordan-Hölder in groups. Recently there have appeared a considerable number of investigations in this field; one should mention the contributions by Birkhoff [2], Klein-Barmen [3, 4], Uzkow [5], George [6], Kurosch [7], and the author [8, 9]. Most of these papers deal particularly with the conditions which it is necessary to impose upon a structure or lattice in order to assert that the chains between two elements have the same length.

In the present paper the general problem of comparing chains in partially ordered sets is considered. It seems remarkable and somewhat surprising that one can formulate a theorem about chains in arbitrary partially ordered sets which contains the theorem of Jordan-Hölder as a special case when applied to more restricted systems. This theorem yields for partially ordered sets, hence by specialization also for structures, a necessary and sufficient condition for the chains to have equal lengths.

2. **Simple cycles.** Let us indicate briefly the terminology which we shall use. The basic partially ordered set to be investigated shall be denoted by P. An element x in P *lies between* two elements $a \supset b$ when $a \supset x \supset b$ and *properly between* them when the two possibilities $a = x$ and $b = x$ are excluded. An element a is *prime over* b and b is *prime under* a when $a \supset b$ and there are no elements properly between a and b. A *chain* is an *ordered subset* of P. A *finite chain* between a and b has the form

$$a = a_0 \supset a_1 \supset \cdots \supset a_{n-1} \supset a_n = b$$

where n is the *length* of the chain. A chain is *complete* when it is not possible to intercalate further terms in it. Through transfinite induction one can prove that any chain is contained in some complete chain.

Now let C_1 and C_2 be two complete chains joining two elements $a \supset b$ in such a manner that they have no elements in common except the end points a and b. Such two chains shall be said to form a *simple cycle* when the following condition is fulfilled:

Simple cycle. There shall exist no elements properly between a and b

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containing or being contained in two different elements c_1 and c_2 belonging to C_1 and C_2, respectively.

The two chains C_1 and C_2 shall be called the *components* of the simple cycle they constitute.

To clarify the definition of a simple cycle let us observe that it is tacitly assumed that c_1 and c_2 are different from the end points a and b of the cycle. The definition clearly excludes that any c_1 in C_1 contains a c_2 in C_2 or conversely.

We shall denote by C_b^a some complete chain from a to b. If C_b^a passes through the two elements $m \supset n$ we shall write C_n^m for the part of the chain connecting m with n. Now let D_n^m be some other complete chain between m and n so that a new chain D_b^a may be formed from C_b^a by replacing C_n^m by D_n^m. When in this case the two chains C_n^m and D_n^m together form the components of a simple cycle we shall say that D_b^a has been obtained from C_b^a by a *simple deformation*. Finally we shall say that two arbitrary chains C_b^a and D_b^a between a and b are *related* and one has been obtained by *deformation* from the other when one can proceed from one to the other by a series of successive simple deformations.

3. **Related chains.** By means of these concepts we can prove the following theorem which is basic for the comparison of chains in partially ordered sets:

THEOREM 1. *Let P be a partially ordered set in which the chains between two elements have a bounded length. Then any two complete chains between two elements $a \supset b$ are related.*

PROOF. The theorem is evidently true when a is prime over b. In this case the longest chain between a and b has the length 1. The theorem may therefore be proved by means of induction with respect to the length of the longest complete chain connecting the two elements $a \supset b$. We assume that the theorem holds for all pairs of elements $a' \supset b'$ for which the maximal chain has a length less than n and prove it for any pair $a \supset b$ for which a maximal chain C_b^a has the length n. We shall have to show that if D_b^a is any other complete chain connecting a and b then D_b^a can be obtained from C_b^a by a series of simple deformations.

One can immediately exclude the possibility of C_b^a and D_b^a having elements in common other than a and b because in this case the desired result follows from the induction assumption. In the case where C_b^a and D_b^a form a simple cycle the result is trivially true. It may therefore be assumed for instance that there exists an element m properly

between a and b such that m contains an element c in C_b^a and an element d in D_b^a. Correspondingly there exist complete chains

$$E_m^a, \quad F_c^m, \quad G_d^m. \tag{1}$$

Since the chain

$$E_m^a + F_c^m \tag{2}$$

joins a to c and since the maximal length between a and c is less than n it follows by the induction assumption that C_c^a and the chain (2) are related. Thus also C_b^a and the chain

$$C' = E_m^a + F_c^m + C_b^c \tag{3}$$

are related. But the two chains

$$F_c^m + C_b^c, \qquad G_d^m + D_b^d \tag{4}$$

are also related as a consequence of the induction so that C' and the chain

$$C'' = E_m^a + G_d^m + D_b^d \tag{5}$$

between a and b are related. Finally one sees that C'' must be related to

$$D_b^a = D_d^a + D_b^d, \tag{6}$$

so that the theorem is proved.

As a referee of this paper Professor S. MacLane observed that Theorem 1 holds also when one assumes only that the chains between any two elements are finite. A proof of this fact will be published subsequently. This gives a more direct proof of Theorem 2.

4. **The main chain theorem.** To obtain the analogue of the Jordan-Hölder theorem that any two complete chains between a and b have the same length it is obviously necessary that the two components in a simple cycle have the same length. But conversely, if this is the case the length of a chain does not change by simple deformations, hence related chains have the same length. From Theorem 1 one concludes therefore that if the chains in the partially ordered set are bounded in length, the necessary and sufficient condition for the Jordan-Hölder theorem to hold is that the two components in a simple cycle have the same length. We shall not formulate this as a theorem since we shall derive the stronger result:

THEOREM 2. *Let P be a partially ordered set in which a chain between*

two elements is always finite. Then the necessary and sufficient condition that any two complete chains connecting two elements $a \supset b$ have the same length is that the two components in a simple cycle have the same length. In this case the two complete chains are always related.

PROOF. The condition is obviously necessary. To prove the sufficiency we observe first that the theorem is true when a is prime over b. Again we can use induction, but this time we shall prefer to use induction with respect to the length of the *shortest* complete chain connecting the two elements $a \supset b$. This procedure corresponds more closely to the ordinary proof of the theorem of Jordan-Hölder in groups and structures. We assume that the theorem holds for all pairs of elements $a' \supset b'$ which can be joined by a complete chain of length less than n and prove it for any pair $a \supset b$ which can be joined by a complete chain C_b^a of length n. We shall show that if D_b^a is any other complete chain connecting a and b then D_b^a can be obtained from C_b^a by a series of simple deformations.

As before we can exclude the case when C_b^a and D_b^a have elements in common other than a and b and also the case where a and b form a simple cycle, since then the result is true by assumption. It may be assumed therefore as previously that there exists an element m properly between a and b such that m contains an element c in C_b^a and an element d in D_b^a. Correspondingly there exist complete chains (1). Since the chain (2) joins a to c and since the chain C_c^a has shorter length than n it follows by the induction assumption that C_c^a and the chain (2) are related. Thus also C_b^a and the chain (3) are related and have the length n. But then the two elements $m \supset b$ are connected by a complete chain of shorter length than n so that the two chains in (4) are related and have the same length according to the induction assumption. Consequently the two chains in (3) and (5) must be related and have the same length n. From this result one concludes finally that the two chains

$$E_m^a + G_d^m, \qquad D_d^a$$

must have a length shorter than n, so that they are related and we have shown that D_b^a and (5) are related and have the length n. This concludes the proof since we have shown successively that the chains C_b^a, C' in (3), C'' in (5) and D_b^a are related and have the same length.

In Theorem 2 the finiteness condition for all chains in P is not absolutely necessary. One can prove the following somewhat more general result:

THEOREM 3. *Let P be a partially ordered set with the property that when at least one of the components in a simple cycle has a finite length then both of them have the same length. If two elements $a \supset b$ in P can be connected by a complete finite chain then all complete chains between a and b are related and have the same length.*

The proof is a simple extension of the proof of Theorem 2. One uses induction with respect to the length n of the shortest complete chain connecting $a \supset b$. With the same notations as before let C_b^a be the finite complete chain of length n between a and b, and D_b^a some complete chain to be compared with C_b^a. As before one can assume that the two chains have no elements in common and that they do not form a simple cycle. Consequently there exists an element m and the corresponding chains (1). By the induction assumption the chain (2) must be finite and related to C_c^a, consequently the chains C_b^a and C' in (3) are related and have the same finite length. By the same argument one passes from C' to C'' in (5) and again from C'' to D_b^a.

5. **The quadrilateral condition.** The theorems derived in the preceding express results on the comparison of chains in partially ordered sets which appear to be about as general as one can expect to obtain them. In most applications where such theorems have been derived they have been obtained under considerably stronger conditions. These conditions are in many cases included in the condition:

Quadrilateral condition. When a and b are two different elements in a partially ordered set, both contained in some element u and both prime over the element d, then there shall exist at least one element m also contained in u and prime over both a and b.

The geometric interpretation of this condition on the basis of the diagram of the partially ordered set is simply that when the two sides (a, d) and (b, d) occur in that part of the diagram which is below u, there shall also exist two other sides (m, a) and (m, b) in the same part of the diagram completing the quadrilateral (a, d, b, m). Theorem 2 gives immediately this result:

THEOREM 4. *Let P be a partially ordered set satisfying the quadrilateral condition. The chains between any two elements $a \supset b$ are supposed to be finite. Then any two complete chains between two elements $a \supset b$ have the same length and one can be obtained from the other by successively replacing two consecutive sides (a_1, a_2) and (a_2, a_3) by two other sides (a_1, b_2) and (b_2, a_3) in a quadrilateral.*

PROOF. Under the given conditions every simple cycle must be a quadrilateral.

It is possible to extend Theorem 4 to partially ordered sets in which the finiteness condition does not hold. It then takes the following form:

THEOREM 5. *Let P be a partially ordered set satisfying the quadrilateral condition. If in P there exist two complete finite chains connecting two elements $a \supset b$ then both chains are related and have the same length.*

PROOF. Except for one additional difficulty one can use the arguments of the ordinary proof of the Jordan-Hölder theorem. Let

$$\begin{aligned} &a \supset a_1 \supset \cdots \supset a_{n-1} \supset a_n = b, \\ &a \supset b_1 \supset \cdots \supset b_{r-1} \supset b_r = b \end{aligned} \tag{7}$$

be the two complete chains connecting a and b. As before we use induction and assume that the theorem is true for all pairs connected by complete chains of length less than n. As a consequence one can assume that $a_{n-1} \neq b_{r-1}$. From the quadrilateral condition follows the existence of an element m contained in a and prime over a_{n-1} and b_{r-1}. Corresponding to (1) we introduce the complete chains

$$E_m^a, \quad F_{a_{n-1}}^m, \quad G_{b_{r-1}}^m$$

where the last two have the length 1. If one can now assume that the chain E_m^a may be taken to be of finite length, the arguments used in the proof of Theorem 1 can be applied to show that both chains in (7) are related and have the same length.

It remains therefore only to establish the existence of a finite complete chain between a and $m = m_{n-2}$. If by chance $m_{n-2} = a_{n-2}$ this is trivial. But when $m_{n-2} \neq a_{n-2}$ there exists by the quadrilateral condition some element m_{n-3} contained in a and prime over a_{n-2} and m_{n-2}. Again the finite chain exists when $m_{n-3} = a_{n-3}$. If this is not the case there is an element m_{n-4} prime over m_{n-3} and a_{n-3}. Clearly the repetition of this process must eventually give a finite chain between a and m.

It should be noted that in Theorem 5 there is no statement about the infinite chains between a and b and the existence of infinite chains is by no means excluded.

6. **Weak quadrilateral condition.** The quadrilateral condition may be weakened in the following manner:

Weak quadrilateral condition. When a and b are distinct elements prime over d there shall exist at least one element m prime over a and b.

This condition is no longer sufficient to prove that any two finite complete chains between two elements $a \supset b$ in the partially ordered set have the same length. It is however sufficient to prove that certain special chains have the same length. This is expressed in the theorem:

THEOREM 6. *Let P be a partially ordered set with a universal element u containing all others and satisfying the weak quadrilateral condition. If there exist two finite complete chains between u and some element a then both chains have the same length and are related.*

The proof may be omitted since it involves only a slight modification of the proof of Theorem 5. For the case where P is a partially ordered set in which all chains from u to arbitrary elements a are finite, the fact that all chains have the same length is a result which has recently been established by Newman [10].

7. **Application to structures.** The preceding theory may now be specialized to the case of structures or lattices. It is clear that in any structure Σ both the quadrilateral condition and the weak quadrilateral condition take the form of the so-called

Birkhoff condition. When a and b are prime over their cross-cut $a \cap b$, the union $a \cup b$ shall be prime over a and b.

When this condition is satisfied in a structure, Theorem 5 gives the main result that finite complete chains between two elements $a \supset b$ are related and have the same length. This is even a little more than the usual formulation for structures satisfying the Birkhoff condition since one ordinarily assumes that *all* chains between two elements are finite.

But the main Theorem 1 and its consequences in Theorems 2 and 3 go so much further that it is possible to use them to formulate a necessary and sufficient condition for the equality of the lengths of chains in a structure. Let us consider first the criterion for a simple cycle in a structure. We denote by C_b^a and D_b^a two complete chains connecting a and b. Then one sees that these two chains are the components of a simple cycle if and only if for every pair of elements c and d different from a and b and belonging to C_b^a and D_b^a, respectively, one has

$$c \cup d = a, \qquad c \cap d = b.$$

This form for the definition of a simple cycle makes it natural to introduce the concept of

Cyclic structure. A structure is cyclic when it consists of two chains

C and D connecting a universal element a with a zero element b such that

$$c \cup d = a, \qquad c \cap d = b$$

for all elements c and d in C and D, respectively.

In other words a cyclic structure is a structure whose elements form a single simple cycle. Let us also say that a cyclic substructure S of a structure Σ is *complete* in Σ when the two chains in Σ defining S are complete. Thus the simple cycles in a structure are the complete cyclic substructures.

We can now state the main theorem on the comparison of the lengths of chains in structures as a direct consequence of Theorem 2:

THEOREM 7. *Let Σ be a structure in which the chains between two elements are finite. The necessary and sufficient condition that all complete chains in Σ between two elements a and b have the same length is that in any complete cyclic substructure the two defining chains always have the same length.*

8. **Application to groups.** To conclude let us mention one further problem which is connected with this theory of chains. The theorem of Jordan-Hölder for *principal chains* (each subgroup in the chain normal in the whole group) in groups follows from the theory of Dedekind structures. The ordinary Jordan-Hölder theorem for *composition series* (each subgroup normal in the preceding) is however not so directly a consequence of the structure theory. This has been observed by various authors and extensions of the abstract theory have been proposed to remedy this deficiency (Uzkow, George, Kurosch). Let us remark that the preceding theory is sufficiently general to solve this difficulty. The partially ordered set of all subgroups occurring in a composition series of the group satisfies the (dual) quadrilateral condition and its complete chains are the composition series in the group, hence they are all related and of the same length. [In detail: If $A \supset B$ are two composition groups, that is, both occur in some composition series, there exists a composition series containing both. When $G \supset B_1 \supset \cdots \supset B_n \supset B \supset \cdots$ is the composition series containing B one obtains the composition series from A to B explicitly as

$$A \supseteq B_1 \cap A \supseteq \cdots \supseteq B_n \cap A \supseteq B.$$

The quadrilateral condition expresses only that if M_1 and M_2 are maximal normal subgroups of A then the intersection $M_1 \cap M_2$ is a maximal normal subgroup both of M_1 and M_2.]

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A CONJECTURE OF ORE ON CHAINS IN PARTIALLY ORDERED SETS

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In a recent investigation, Ore[1] has given a form of the Jordan-Hölder theorem valid for an arbitrary partially ordered set P. This theorem involves essentially the deformation of one chain into another by successive steps, each step being like that used in the conventional Jordan-Hölder theorem. Ore observes that his first theorem would be slightly easier to apply if it were proved under a weaker hypothesis. The modified theorem runs as follows:[2]

THEOREM. *If P is a partially ordered set in which every chain joining two elements is finite, then any complete chain between two elements $b<a$ can be deformed into any other complete chain between the same two elements.*

The proof rests on this lemma:

LEMMA. *Under the hypothesis of the theorem, if C is a complete chain from b to a which cannot be deformed into the complete chain D from b to a, there exist in P elements $b'<a'$ and complete chains C' and D' from b' to a' such that C' cannot be deformed into D' and such that $b\leqq b'$, $a'\leqq a$ where either $b<b'$ or $a'<a$.*

PROOF. *Case* 1. C and D have in common the element e, $b<e<a$. Then either C_b^e cannot be deformed into D_b^e, or C_e^a cannot be deformed into D_e^a. In these two cases, set $b'=b$, $a'=e$ or $b'=e$, $a'=a$, respectively.

Case 2. C and D have no elements in common. Since C cannot be deformed into D, they cannot together constitute a simple cycle. There will then exist, say, elements c in C and d in D with $b<c<a$, $b<d<a$ and an element m in P with $c\leqq m<a$, $d\leqq m<a$. Because of the hypothesis that every chain in P joining two elements is finite, there will exist in P finite complete chains E_m^a, F_c^m, G_d^m. Then b is joined to a by four complete chains,

$$C_b^c+C_c^a, \qquad C_b^c+F_c^m+E_m^a,$$
$$D_b^d+G_d^m+E_m^a, \qquad D_b^d+D_d^a.$$

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[1] Oystein Ore, *Chains in partially ordered sets*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 558–566.

[2] Terminology and notation follow the paper of Ore.

Since C cannot be deformed into D, one of the following three deformations must be impossible:

$$C_c^a \to F_c^m + E_m^a, \qquad C_b^c + F_c^m \to D_b^d + G_d^m,$$
$$G_d^m + E_m^a \to D_d^a.$$

In the first case we set $a' = a$, $b' = c$; in the second case, $a' = m$, $b' = b$; in the third case $a' = a$, $b' = d$. In each case we have the conclusion of the lemma.

To prove the theorem, suppose that P were to contain two complete chains C and D joining b to a in such wise that C cannot be deformed into D. By induction on n, the lemma gives in P elements $a = a_0 \geqq a_1 \geqq \cdots \geqq a_n$ and $b = b_0 \leqq b_1 \leqq b_n \leqq a_n$ such that for each i either $a_{i-1} > a_i$ or $b_{i-1} < b_i$ $(i = 1, \cdots, n)$, and such that there are complete chains C_n, D_n joining b_n to a_n with C_n not deformable into D_n. This construction can be carried on indefinitely, using the axiom of choice to select at each stage a definite pair a_{n+1}, b_{n+1}. This produces two sequences of elements a_i, b_i with

$$b_0 \leqq b_1 \leqq b_2 \leqq \cdots \leqq \cdots \leqq a_2 \leqq a_1 \leqq a_0.$$

Furthermore, the inequality sign holds an infinite number of times here, so that we obtain an infinite chain joining $b = b_0$ to $a = a_0$, contrary to the hypothesis of the theorem.

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ON IRREDUCIBLE CONTINUOUS CURVES

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This paper deals with the existence of continuous curves containing compact and closed point sets and with certain properties of continuous curves which are irreducible continua about point sets in spaces which are not necessarily metric. Previous results on these topics have been almost entirely for metric spaces. Thus Gehman[1] proved for the plane that, given a compact continuum, there exists a compact continuous curve, which is the sum of a countable number of arcs plus its limit points, containing it. Whyburn and Ayres[2] extended this to a space of a continuous curve in n dimensions, and Zippin[3] indicated that their argument might be modified to give the following: If T is a closed and compact subset of a complete metric continuous curve S, there exists a compact continuous curve which is a subset of S and contains T. Zippin[4] proved that, given a complete metric continuous curve S and a compact, closed, one-dimensional subset, T, of S, such that every component of T is a continuous curve, and no more than a finite number of components of T are of diameter greater than any positive number, there exists a compact continuous curve which is a subset of S and an irreducible continuum about T. Miss Miller[5] proved that if, in a connected space satisfying Axioms $0-2$ of R. L. Moore's *Foundations of point set theory*,[6] T be a compact and closed point set, there exists a compact continuum containing T.

The concept of a continuum irreducible about a subset of itself was first introduced by Wilson.[7] Most of the past work on continuous curves which are irreducible continua about point sets has been done

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[1] H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Ann. of Math. (2) vol. 27 (1926) p. 30.

[2] G. T. Whyburn and W. L. Ayres, *On continuous curves in n dimensions*, Bull. Amer. Math. Soc. vol 34 (1928) p. 350.

[3] L. Zippin, *On continuous curves irreducible about subsets*, Fund. Math. vol. 20 (1933) pp. 197–205.

[4] L. Zippin, loc. cit.

[5] H. C. Miller, *A theorem concerning closed and compact point sets which lie in connected domains*, Bull. Amer. Math. Soc. vol. 46 (1940) p. 848.

[6] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications vol. 13, New York, 1932.

[7] W. A. Wilson, *On the oscillation of a continuum at a point*, Trans. Amer. Math. Soc. vol. 27 (1925) p. 433.

by Gehman, who obtained results[8] equivalent to the following theorem for the plane: If M be a compact irreducible continuous curve about a closed point set K, then (1) M is an irreducible continuum about K; (2) K contains all the non-cut points of M; (3) if H is a subcontinuum of M, H is connected im kleinem at every point of $H(M-K)$. He[9] gave a later proof of part (2) which with very slight modification shows that if, in a space satisfying Axioms $0-1$, M be a locally compact continuous curve which is an irreducible continuum about a closed subset K of M, K contains all the non-cut points of M. Zippin[10] indicated a proof of part (1) if M be a complete metric space and K be compact.

THEOREM 1. *In a connected space satisfying Axioms* $0-2$, *every closed and compact point set* T *which has no continuum of condensation is a subset of a compact hereditary continuous curve.*

PROOF. Every component of T is a continuous curve and hence is arcwise connected. By Axiom 2 regions may be considered as connected domains. There exists[11] an infinite sequence $W_1, W_2, W_3, \cdots$ such that (1) for each n W_n is a finite subcollection of G_n covering T; (2) if g be a member of W_{n+1}, $\bar{g}$ is a subset of some member of W_n; (3) if H and K be two mutually exclusive closed subsets of T, and k be a positive integer, there exists a positive integer m such that if U be a coherent collection of k regions, each belonging to a member of the sequence $W_m, W_{m+1}, \cdots$, and U^* contain a point of H,[12] then U^* contains no point of K; (4) every region of each W_n contains a point of T, and if $P_1, P_2, \cdots$ be a sequence of points such that for each n P_n is a subset of W_n, then some subsequence of that sequence converges to a point of T. For each region R_{i_1} in W_1 let P_{i_1} be a point of $R_{i_1} \cdot T$. Let α_{i_1} denote an arc from P_{1_1} to P_{i_1} which is constructed to lie in the component of T containing P_{1_1} if that component contains P_{i_1}. Let α_1 be the sum of all α_{i_1}. For each pair of regions, R_{i_1} of W_1 and R_{i_2} of W_2, such that R_{i_1} contains R_{i_2}, let $P_{i_1 i_2}$ be a point of $R_{i_2} \cdot T$. Let $\alpha_{i_1 i_2}$ denote an arc from P_{i_1} to $P_{i_1 i_2}$ lying in R_{i_1} or in the component of T which contains P_{i_1} according as that component does not or does contain $P_{i_1 i_2}$. Let α_2 be the sum of α_1 and all $\alpha_{i_1 i_2}$. For each

[8] H. M. Gehman, *Irreducible continuous curves*, Amer. J. Math. vol. 49 (1927) pp. 189–196.

[9] H. M. Gehman, *Concerning certain types of non-cut points, with an application to continuous curves*, and *Concerning irreducible continua*, Proc. Nat. Acad. Sci. U.S.A. vol. 14 (1928) pp. 431–435.

[10] L. Zippin, loc. cit.

[11] R. L. Moore, loc. cit. pp. 58–59, Theorem 81.

[12] The point set U^* is the sum of all point sets of the collection U.

triplet of regions, R_{i_1} of W_1, R_{i_2} of W_2, and R_{i_3} of W_3, such that R_{i_1} contains R_{i_2} which contains R_{i_3}, let $P_{i_1i_2i_3}$ be a point of $R_{i_3} \cdot T$. Let $\alpha_{i_1i_2i_3}$ denote an arc from $P_{i_1i_2}$ to $P_{i_1i_2i_3}$ lying in R_{i_2} or in the component of T which contains $P_{i_1i_2}$ according as that component does not or does contain $P_{i_1i_2i_3}$. Let α_3 be the sum of α_2 and all $\alpha_{i_1i_2i_3}$. Continue the indicated constructions indefinitely. Let α be the sum of all α_n.

The point set $\bar{\alpha}$ is a hereditary continuous curve containing T. First, since α is the sum of a monotonic sequence of connected sets, it is connected, so that $\bar{\alpha}$ is a continuum. Second, $\bar{\alpha}$ contains T because, given a point P of T and a region R containing P, there exists for some n a region R_{i_n} of W_n which is a subset of R and contains P; there then exists a monotonic sequence of regions R_{i_1}, R_{i_2}, $\cdots$, R_{i_n}, so that R_{i_n} contains a point $P_{i_1 \cdots i_n}$ of α; whence R contains a point of α and P is a limit point of α.

Suppose that α contains an infinite point set K such that $\alpha + T$ contains no limit point of K. For each n, $\alpha_n + T$ is compact and therefore contains only a finite number of points of K. Since W_n covers $\alpha - (\alpha_n + T)$, $W_n^* \cdot K$ is infinite. This contradicts part (4) of the condition on the W_n's. It follows that $\bar{\alpha}$ is compact and that T contains $\bar{\alpha} - \alpha$.

It remains to prove that every subcontinuum of $\bar{\alpha}$ is a continuous curve. Suppose that this is not true. Then there exists a subcontinuum M of $\bar{\alpha}$, containing a point P at which it is not connected im kleinem. Suppose, first, that P is not a point of T. By part (4) of the condition on the W_n's there exists an integer k such that no region W_{k-1} contains P; by part (2), if R_{i_k} is a region of W_k, $\overline{R}_{i_k}$ does not contain P. As W_k is a finite collection there exists a region R containing P that contains no point of W_k^* and so contains no point of T. Hence $R \cdot \bar{\alpha}$ is a subset of $R \cdot \alpha_k$. As R contains the point P at which M is not connected im kleinem, it contains a continuum of condensation of M and so of α_k, the sum of a finite number of arcs. This is impossible.

Suppose now that P is a point of T. There exist a domain D containing P, and a sequence of mutually exclusive continua K, M_1, M_2, $\cdots$ converging to K, such that (1) K contains P; (2) every continuum of the sequence contains a point of D and a point of $\overline{D} - D$; (3) each M_j is a component of $M \cdot \overline{D}$. Since it contains no point where M is connected im kleinem, $K \cdot D$ is a subset of T. If every point of $K \cdot D$ is a limit point of $T - K \cdot D$, then if D' be a domain intersecting K, $\overline{D}'$ a subset of D, some component of $K \cdot \overline{D}'$ will be nondegenerate and a continuum of condensation of T. Hence

some point P' of $K \cdot D$ is not a limit point of $T-K \cdot D$. Let D' be a domain containing P', such that $\overline{D}'$ is a subset of D and contains no point of $T-K \cdot D$. Let D'' be a domain containing P' such that $\overline{D}''$ is a subset of D'. Let D''' be a domain containing P' such that $\overline{D}'''$ is a subset of D''. Let K' be the component of $K \cdot \overline{D}'''$ containing P'. K' is a nondegenerate subcontinuum of T and a subset of the limiting set of the sequence $M_1 \cdot \overline{D}''$, $M_2 \cdot \overline{D}''$, $\cdots$. By parts (3) and (4) of the conditions on the W_n's, there exists an integer k such that no region of W_n, $n \geqq k$, intersects $\overline{D}''$ and $S-D'$. $\alpha_{i_1 \cdots i_n}$ lies in $R_{i_{n-1}}+T$, so every arc $\alpha_{i_1 \cdots i_n}$, $n>k$, which intersects $\overline{D}''$ and $S-D'$ lies in T. Hence every arc $\alpha_{i_1 \cdots i_n}$, $n>k$, which intersects $\overline{D}''$ lies in D' or in T; both end points of such an arc lie in T, by its construction, so that either the whole arc lies in T or its end points lie in $T \cdot D'$ and so in K; in the latter case the end points lie in the same component of T and so the arc lies in T. Therefore $\bar{\alpha} \cdot D''$ is a subset of α_k+T.

It follows that $M_1 \cdot \overline{D}''$, $M_2 \cdot \overline{D}''$, $\cdots$ are all subsets of α_k+T. Therefore K' is a continuum of condensation of α_k+T. $K' \cdot (\overline{T-K'})$ and $K' \cdot (\overline{\alpha_k-\alpha_k \cdot K'})$ are closed and totally disconnected. But their sum is the continuum K'. This involves a contradiction.

THEOREM 2. *In a connected space satisfying Axioms* $0-2$, *every closed and compact point set* T *is a subset of a compact continuous curve.*

PROOF. By Axiom 2 regions may be considered as connected domains. Construct the W_n's and the $P_{i_1 \cdots i_n}$'s exactly as in the proof of Theorem 1. Define the $\alpha_{i_1 \cdots i_n}$'s as follows: α_{i_1} is an arc from P_1, to P_{i_1}; $\alpha_{i_1 \cdots i_n}$ is an arc from $P_{i_1 \cdots i_{n-1}}$ to $P_{i_1 \cdots i_n}$ in $R_{i_{n-1}}$. Define α_n and α as before on this basis. It will be proved that $\bar{\alpha}$ is a compact continuous curve containing T.

By the argument used in the proof of Theorem 1 it may be shown that $\bar{\alpha}$ is a compact continuum containing T. It remains to show that $\bar{\alpha}$ is connected im kleinem at every point. Let P be any point of $\bar{\alpha}$, and R a region containing P. There exists a positive integer k such that if R_1 and R_2 are any two intersecting regions of W_k, one containing P, then R contains $\overline{R}_1+\overline{R}_2$. For each set of indices $i_1, i_2, \cdots, i_k$ let $L_{i_1 \cdots i_k}$ be the sum of all $\alpha_{i_1 \cdots i_n}$'s, $n>k$, whose first k indices are the members of that set. The set of $L_{i_1 \cdots i_k}$'s is finite. If α_k, the sum of a finite number of arcs, contains P, it is connected im kleinem there. Let R' be a region containing P such that (1) if P' is a point of $\alpha_k \cdot R'$, there is a connected subset of $\alpha_k \cdot R$ containing P and P'; (2) if R' contains a point of $\overline{L_{i_1 \cdots i_k}}$, $\overline{L_{i_1 \cdots i_k}}$ contains P. Let Q be any

point of $R' \cdot \bar{\alpha}$. If Q is a point of α_k, there is a connected subset of $R \cdot \bar{\alpha}$ containing P and Q. Suppose it does not belong to α_k. Then it must belong to some $L_{i_1 \cdots i_k}$. Then that $L_{i_1 \cdots i_k}$ contains P and is a subset of $\overline{R}_{i_k}$. Since $\overline{R}_{i_k}$ contains P it is a subset of R. Hence $L_{i_1 \cdots i_k}$ is a subset of $R \cdot \bar{\alpha}$ which is plainly connected and contains P and Q. The theorem is therefore proved.

Theorem 2 does not remain true if "locally compact" replaces "compact." For let space be the sum of the intervals in the plane from (0, 0) to (1, 0) and from $(1/n, 0)$ to $(1/n, 1)$, $(n>0)$, and let T be the set of points $(1/n, 1)$. Here space is a locally non-compact continuous curve which is an irreducible continuum about T. Furthermore, there is a non-cut point of space not belonging to T.

THEOREM 3. *If space satisfies Axioms* $0-1$, *and* M *is an irreducible continuous curve about a compact and closed subset* T *of* M, M *is a compact irreducible continuum about* T.

PROOF. By Theorem 2, M contains a compact continuous curve U containing T. Since it is identical with U, M is compact.

Suppose that there exists a proper subcontinuum V of M containing T. Let P be a point of $M-V$, and let C be the component of $M-P$ containing V. The set C is a domain with respect to M and hence considered as space satisfies Axioms $0-2$. Hence, by Theorem 2, C contains a compact continuous curve U' containing T. This is a contradiction.

THEOREM 4. *If space satisfies Axioms* $0-1$, *and* M *is a locally compact continuous curve which is an irreducible continuum about a closed subset* T *of* M, *every continuum of condensation of* M *is a continuum of condensation of* T.

PROOF. The following lemma must first be demonstrated:

LEMMA. *Under the conditions of Theorem* 4, *every continuum of condensation of* M *is a subset of* T.

PROOF OF LEMMA. Regard M as space. Then Axiom 2 holds. Suppose that M has a continuum of condensation N which does not lie wholly in T. Since M is locally compact, N contains a nondegenerate compact continuum N' which lies in $M-T$. N' contains a subcontinuum N'' which is an irreducible continuum between two points A and B. Every component of $M-N''$ intersects T since its complement is a continuum. Since M is locally compact, N'' is a subset of a domain D such that $\overline{D}$ is a compact subset of $M-T$. The set $\overline{D}-D$ is the sum of its intersections with the components of $M-N''$, and

each component intersects it since the component has a limit point in N'' and in T. Since no closed and compact point set is the sum of infinitely many mutually exclusive open subsets of itself, $M-N''$ has only a finite number of components. As every point of N'' is a cut point of M,[13] every non-cut point of N'' is the boundary of some component of $M-N''$, so the set of all non-cut points of N'' is finite. There exists a component K of $M-N''$ such that $\overline{K}\cdot N''$ is infinite. There then exist three points of $\overline{K}\cdot N''$ which lie on the segment AB of N'', such that no one of them is the entire boundary of any component of $M-N''$. Let P be that one of them which lies between the other two on the segment AB of N''.[14] Then $K+N''-P$ is plainly connected. Since $N''-P$ contains a limit point of every component of $M-N''$, $M-P$ is connected, which leads to a contradiction.

Suppose now that Theorem 4 is not true. Then, by the Lemma, T contains a continuum N which is a continuum of condensation of M but not of T. Let P be a point of N not belonging to $\overline{T-N}$. Let Q be a point of $N-P$. Let C be any subcontinuum of M which contains $\overline{T-N}+Q$. Since it contains T, the continuum $C+N$ is M. Hence C contains $M-N$, and therefore N, so that C is M and M is an irreducible continuum about $\overline{T-N}+Q$. As N contains a point P not in $\overline{T-N}+Q$, this contradicts the Lemma, and the theorem is proved.

THEOREM 5. *In a connected space satisfying Axioms* $0-2$, *every compact and closed point set* T *with no continuum of condensation is a subset of a compact continuous curve which has no continuum of condensation.*

PROOF. By Theorem 1 there exists a compact hereditary continuous curve K containing T. The set K contains a continuum M which is an irreducible continuum about T. Since M is a continuous curve, by Theorem 4 M has no continuum of condensation.

In conclusion I wish to express my deepest thanks to Professor R. L. Moore for invaluable guidance in the preparation of this paper.

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[13] H. M. Gehman, loc. cit.

[14] R. L. Moore, loc cit. pp. 35 ff.

NEW SYSTEMS OF HYPERGEODESICS DEFINED ON A SURFACE

P. O. BELL

Introduction. Let a non-ruled surface S be referred to its asymptotic net as parametric. As a point P_y moves along a curve C_λ of S, the tangents at P_y to the u- and v-asymptotic curves of S describe two ruled surfaces R_λ^u and R_λ^v, respectively. Let S_ρ and S_σ denote arbitrary transversal surfaces of the congruences of u- and v-tangents of S, respectively. The purpose of the present paper is to introduce and study systems of curves of S which will be called ρ- and *σ-tangeodesics.*

DEFINITION. *A curve C_λ of S whose associated ruled surface R_λ^u intersects the surface S_ρ in an asymptotic curve of R_λ^u is a ρ-tangeodesic of S. Similarly, a curve C_λ of S whose associated ruled surface R_λ^v intersects S_σ in an asymptotic curve of R_λ^v is a σ-tangeodesic of S.*

The ρ- and *σ-tangeodesics* of S at P_y are found to be associated in remarkable manners with the *edges of Green*, the *directrices of Wilczynski*, and the *projective normal of Fubini*. In fact, a new geometric characterization is obtained for each of these lines.

1. **Tangeodesics.** If the parametric net on a non-ruled surface S is the asymptotic net, the homogeneous projective coordinates $y^{(i)}(u, v)$ $(i=1, 2, 3, 4)$ of a general point P_y of S are solutions of a system of differential equations which may be assumed to be reduced to Wilczynski's canonical form

$$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a'y_u + gy = 0. \tag{1.1}$$

The homogeneous coordinates of points ρ, σ on arbitrarily selected transversal surfaces S_ρ and S_σ of the congruences of u- and v-tangents of S are given by the vector forms

$$\rho = y_u - \beta y, \qquad \sigma = y_v - \alpha y, \tag{1.2}$$

wherein β, α are arbitrary analytic functions of u, v.

Let l denote the line joining ρ, σ and let l' denote its reciprocal at P_y. The line l' joins the points P_y and z where z is given by

$$z = y_{uv} - \alpha y_u - \beta y_v \tag{1.3}$$

in which β and α are the functions in (1.2). The line l, according to Green's classification, is a line of the first kind and generates a con-

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gruence Γ of the first kind as P_y moves over S. The line l' is a line of the second kind and generates a congruence Γ' of the second kind as P_y moves over S.

Let C_λ denote an integral curve of the curvilinear differential equation

$$dv - \lambda(u, v)du = 0. \tag{1.4}$$

Regarding u as independent variable we write $v' = \lambda(u, v)$ and $v'' = \lambda_u + \lambda\lambda_v$, in which accents indicate differentiation with respect to u.

The homogeneous coordinates of a general point of the ruled surface R_λ^u are represented by the vector form

$$\bar{y} = y_u + wy, \tag{1.5}$$

wherein u and w are independent variables and v varies in accordance with the relation $v' = \lambda(u, v)$.

Let us put $u = u(t)$, $w = w(t)$, so that $\bar{y}$ describes a curve on R_λ^u as t varies. The necessary and sufficient condition that this curve be an asymptotic curve of R_λ^u is that the determinant equation

$$(\bar{y}, \bar{y}_u + v'\bar{y}_v, \bar{y}_w, d^2\bar{y}/dt^2) = 0 \tag{1.6}$$

be satisfied. If we transform equation (1.6) by making use of equations (1.5), (1.4) and (1.1) we obtain, in view of the inequality $(y, y_u, y_v, y_{uv}) \neq 0$, the equation

$$dw/du = \{2b^2 + (b_u - 2bw)v' + (w^2 + 2b_v + f)v'^2 - 2a'bv'^3 - bv''\}/v'^2. \tag{1.7}$$

As P_y moves along C_λ the point ρ moves in the direction defined by (1.7) if and only if $w = -\beta$ satisfies (1.7). To obtain, therefore, the curvilinear differential equation for the ρ-tangeodesics we have merely to substitute $-\beta$ for w in (1.7) and clear of fractions. The result is

$$bv'' - 2b^2 - (2b\beta + b_u)v' - (\beta^2 + 2b_v + f + \beta_u)v'^2 + (2a'b - \beta_v)v'^3 = 0. \tag{1.8}$$

The differential equation for the σ-tangeodesics may be obtained by making the substitution

$$\begin{pmatrix} v'' & v' & v & b & a' & \beta & f \\ -v''/v'^3 & 1/v' & u & a' & b & \alpha & g \end{pmatrix}$$

in (1.8). The result, on simplifying, is

$$a'v'' + \alpha_u - 2a'b + (\alpha^2 + 2a_u' + g + \alpha_v)v' + (2a'\alpha + a_v')v'^2 + 2a'^2v'^3 = 0. \tag{1.9}$$

2. **Systems of hypergeodesics which have no cusp-axes.** The curves defined on a surface S by a differential equation of the form

$$v'' = A + Bv' + Cv'^2 + Dv'^3 \tag{2.1}$$

in which the coefficients are functions of u, v and accents indicate differentiation with respect to the independent variable u, are called *hypergeodesics.*[1] *The envelope of the osculating planes at a point P_y of the hypergeodesics* (2.1) *is a cone which is ordinarily of the third class.* When this cone is of the third class it has three distinct cusp-planes which intersect in a line called *the cusp-axis of the cone*, or *the cusp-axis of the hypergeodesics at the point P_y*. The cusp-axis is the line l' for which α and β are given by

$$\alpha = C/2, \qquad \beta = -B/2. \tag{2.2}$$

We are interested here in those cases in which the class of the cone is less than three and the cone has no cusp-axis. The local equation of the osculating plane at P_y of the curve C_λ defined by (1.4) is

$$2\lambda(\lambda x_2 - x_3) + (\lambda' - 2b + 2a'\lambda^3)x_4 = 0, \tag{2.3}$$

when referred to the tetrahedron whose vertices have the general coordinates y, y_u, y_v, y_{uv}. Assuming C_λ to be an integral curve of (2.1) we replace λ' by the right member of (2.1) and put λ in place of v'. The result is

$$2\lambda(\lambda x_2 - x_3) + (A - 2b + B\lambda + C\lambda^2 + [D + 2a']\lambda^3)x_4 = 0. \tag{2.4}$$

The *union curves of a congruence* Γ' form a system of hypergeodesics, sometimes called an *axial system*, whose osculating planes at P_y all pass through the line l' of the congruence Γ'. Equation (2.1) represents such a system if $A = 2b$ and $D = -2a'$.

A system of hypergeodesics (2.1) for which

$$A = 2b, \qquad D \neq -2a' \tag{2.5}$$

will be called, for reasons which appear later, a *u-polar system*; and a system (2.1) for which

$$D = -2a', \qquad A \neq 2b \tag{2.6}$$

will be called a *v-polar system.*

If system (2.1) is a u-polar system, the equation for the envelope of its osculating planes at P_y may be readily found from (2.4) to be

$$(2x_2 + Cx_4)^2 - 4(D + 2a')(Bx_4 - 2x_3)x_4 = 0. \tag{2.7}$$

Similarly, if (2.1) represents a v-polar system the equation for the envelope of its osculating planes at P_y may be found to be

[1] G. Fubini, *Fondamenti della geometria proiettivo-differenziale di una superficie*, Atti Accad. Sci. Torino vol. 53 (1918) p. 1034.

$$(2.8) \qquad (2x_3 - Bx_4)^2 - 4(A - 2b)(2x_2 + Cx_4)x_4 = 0.$$

Since the cones (2.7) and (2.8) are nondegenerate quadric cones, they have no cusp-axes at P_y. Hence we have that *neither a u-polar system nor a v-polar system of hypergeodesics has a cusp-axis at* P_y.

There are two generators of the cone (2.7) which are such that the tangent planes of the cone along these generators pass through the u-tangent to S at P_y. One of these is the v-tangent to S at P_y and the other is the line l' for which

$$(2.9) \qquad \alpha = C/2, \qquad \beta = -B/2,$$

wherein B and C are the functions appearing in (2.7). This line l' will be called the *u-edge* of the *u-polar system*.

The *v-edge* of a *v-polar system* is characterized similarly.

Since equations (2.9) are of the same form as equations (2.2), we have immediately this theorem.

THEOREM 2.1. *If the coefficients B and C of the differential equation of a non-polar system of hypergeodesics are identical with the corresponding coefficients of the differential equation of a u-polar system of hypergeodesics, the cusp-axis of the non-polar system at* P_y *coincides with the u-edge of the u-polar system at* P_y.

A similar theorem applies, of course, to a v-polar system of hypergeodesics.

The forms of the differential equations (1.8) and (1.9) show clearly that the ρ- and σ-tangeodesics form u- and v-polar systems of hypergeodesics. For the system (1.8) we have

$$(2.10) \qquad A = 2b, \quad B = 2\beta + b_u/b, \quad C = (\beta^2 + 2b_v + f + \beta_u)/b, \quad D = (\beta_v - 2a'b)/b.$$

For the system (1.9) we have

$$(2.11) \qquad A = (2a'b - \alpha_u)/a', \quad B = -(\alpha^2 + 2a_u' + g + \alpha_v)/a', \quad C = -2\alpha - a_v'/a', \quad D = -2a'.$$

The cone (2.7) is associated with the system (1.8) of ρ-tangeodesics if A, B, C, D are given by (2.10). Similarly, if A, B, C, D are defined by (2.11), the cone (2.8) is associated with the system (1.9) of σ-tangeodesics.

The *u-edge* of the *ρ-tangeodesics* (1.8) is the line l_1' passing through the points P_y and z_1, where z_1 is given by $z_1 = y_{uv} - \alpha_1 y_u - \beta_1 y_v$, in which

$$(2.12) \qquad \beta_1 = -\beta - b_u/2b, \quad \alpha_1 = (\beta^2 + 2b_v + f + \beta_u)/2b.$$

The v-edge of the σ-tangeodesics is the line l_2' passing through P_y and z_2 where z_2 is given by $z_2 = y_{uv} - \alpha_2 y_u - \beta_2 y_v$, in which

$$(2.13)\quad \beta_2 = (\alpha^2 + 2a_u' + g + \alpha_v)/2a', \quad \alpha_2 = -\alpha - a_v'/2a'.$$

3. The edges of Green, the directrices of Wilczynski, and the projective normal. Let us apply the results of §§1 and 2 to obtain new characterizations of the *edges of Green*, the *directrices of Wilczynski*, and the *projective normal of Fubini*. The plane which is tangent to the cone (2.7) of the ρ-tangeodesics along the *u-edge* intersects the plane which is tangent to the cone (2.8) of the σ-tangeodesics along the *v-edge* in a line $\bar{l}'$ of the second kind which will be called the *joint-edge of the systems of ρ- and σ-tangeodesics of S at P_y*. This line passes through the points P_y and $\bar{z}$ where the general coordinates of $\bar{z}$ are given by $\bar{z} = y_{uv} - \bar{\alpha} y_u - \bar{\beta} y_v$, in which

$$(3.1)\qquad \bar{\alpha} = -\alpha - a_v'/2a', \qquad \bar{\beta} = -\beta - b_u/2b.$$

Since the functions α, β associated with the edges of Green are given by

$$(3.2)\qquad \alpha = -a_v'/4a', \qquad \beta = -b_u/4b,$$

we have the following theorem.

THEOREM 3.1. *The second edge of Green at P_y lies in the plane π determined by the joint-edge of the systems of ρ- and σ-tangeodesics of S at P_y and the reciprocal l' of the line l joining ρ, σ. The joint-edge coincides with the line l' if and only if l' is the second edge of Green. Any two particular planes π_1 and π_2 of the plane π (corresponding to selections ρ_1, σ_1 and ρ_2, σ_2) intersect in the second edge of Green.*

Let σ_a denote the intersection of the tangent plane to S_ρ at ρ with the v-tangent to S at P_y and let ρ_a denote the intersection of the tangent plane to S_σ at σ with the u-tangent to S at P_y. It may be easily verified that the general coordinates of ρ_a and σ_a are given by $\rho_a = y_u - \beta_a y$, $\sigma_a = y_v - \alpha_a y$, wherein β_a, α_a are given by

$$(3.3)\qquad \beta_a = -(g + \alpha_v + \alpha^2)/2a', \quad \alpha_a = -(f + \beta_u + \beta^2)/2b.$$

The line l_a joining ρ_a, σ_a was introduced by the author in a previous paper[2] and called *the asymptotic associate of the line l* joining ρ, σ.

The plane determined by the v-tangent of S at P_y and the u-edge of the ρ-tangeodesics at P_y is the polar plane of the u-tangent of S at P_y with respect to the cone (2.7) of the ρ-tangeodesics. Similarly, the

[2] P. O. Bell, *A study of curved surfaces by means of certain associated ruled surfaces*, Trans. Amer. Math. Soc. vol. 46 (1939) p. 396.

plane determined by the u-tangent of S at P_y and the v-edge of the σ-tangeodesics at P_y is the polar plane of the v-tangent of S at P_y with respect to the cone (2.8) of the σ-tangeodesics. These two polar planes intersect in a line l_3' which will be called the *polar-axis of the ρ- and σ-tangeodesics at P_y*. This line may be shown to pass through the points P_y and z_3 where the general coordinates of z_3 are given by $z_3 = y_{uv} - \alpha_3 y_u - \beta_3 y_v$, in which

$$\alpha_3 = -\alpha_u + b_v/b, \qquad \beta_3 = -\beta_u + a_u'/a'. \tag{3.4}$$

Since the functions α, β for the directrix l' of Wilczynski are given by

$$\alpha = b_v/2b, \qquad \beta = a_u'/2a', \tag{3.5}$$

equations (3.4) are such that we have, immediately, this theorem.

THEOREM 3.2. *The second directrix of Wilczynski lies in the plane p determined by the polar-axis of the ρ- and σ-tangeodesics at P_y and the reciprocal l_3', with respect to S at P_y, of the asymptotic associate of l. Any two particular planes p_1 and p_2 of the plane p (corresponding to lines l_1 and l_2) intersect in the second directrix of Wilczynski.*

Theorems 3.1 and 3.2 may be dualized by replacing the lines and planes of these theorems by their reciprocals with respect to S at P_y. The dual of Theorem 3.1 is the following theorem.

THEOREM 3.3. *The first edge of Green contains the point P of intersection of the reciprocal of the joint-edge of the systems of ρ- and σ-tangeodesics of S and the line l joining ρ, σ. These three lines coincide if and only if the line l is the first edge of Green. Any two particular points P_1 and P_2 of the point P (corresponding to selections ρ_1, σ_1 and ρ_2, σ_2) determine the first edge of Green.*

The statement of the dual of Theorem 3.2 will be left to the care of the reader.

Finally, since the *projective-normal* of S at P_y is the line for which the functions α, β are given by $\alpha = -(b_v/2b + a_v'/2a')$, $\beta = -(a_u'/2a' + b_u/2b)$, and the first directrix of S at P_y is the line l for which $\alpha = b_v/2b$, $\beta = a_u'/2a'$, we have from equations (3.1) this theorem.

THEOREM 3.4. *If the line l joining ρ, σ is the first directrix of Wilczynski, the joint-edge of the systems of ρ and σ-tangeodesics of S at P_y is the projective normal of Fubini.*

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ZEROS AND POLES OF FUNCTIONS DEFINED BY TAYLOR SERIES

MICHAEL GOLOMB

1. **Introduction.** With the objective of providing a straightforward numerical method for the determination of poles and zeros of functions defined by Taylor series this note reexamines Hadamard's solution of this problem, which is found in his classical thesis.[1] The best known part of Hadamard's solution is the criterion which enables one to determine the meromorphic character of the expanded function and the total number of poles on the circle of convergence. But this solution also includes a method of determining these poles as functions of the Taylor coefficients, and Hadamard himself intimated that his results should prove useful in the numerical evaluation of poles and zeros. However, it seems that, as a device in numerical analysis, his method has attracted much less attention than it deserves. This may be due to the fact that Hadamard's criterion for the number of poles employs limits superior, which are impractical for numerical work.

In this paper no use is made of limits superior, and the number of poles on the circle of convergence is ascertained by the process of evaluating their affixes. Besides determining the polynomial whose zeros are all the poles of the expanded function on the circle of convergence with their proper multiplicities, the paper also determines the polynomial whose zeros are the different poles of highest order only. These results are based on an identity and an inequality for persymmetric determinants involving successive Taylor coefficients of rational and meromorphic functions, which seem to be new, and may also prove useful in other applications.

2. **A formula for persymmetric determinants.** We first are going to establish an identity for persymmetric determinants of the form

$$(1)\qquad d_n^{(m)} = \begin{vmatrix} c_{n+1} & c_{n+2} & \cdots & c_{n+m} \\ c_{n+2} & c_{n+3} & \cdots & c_{n+m+1} \\ \cdot & \cdot & \cdots & \cdot \\ c_{n+m} & c_{n+m+1} & \cdots & c_{n+2m-1} \end{vmatrix}, \qquad n = -1, 0, 1, \cdots,$$

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[1] J. Hadamard, *Essai sur l'étude des fonctions données par leur développement de Taylor*, J. Math. Pures Appl. (4) vol. 8 (1892) pp. 101–186.

where the $2m-1$ numbers $c_{n+1}, c_{n+2}, \cdots, c_{n+2m-1}$ are successive coefficients in the Taylor series expansion of a rational function.

Suppose, at first, the rational function has only simple poles and is of the form $g_0(z)=\sum_{n=0}^{\infty}c_nz^n=-\sum_{k=1}^{p}\zeta_k/(z-z_k)$, $z_k\neq 0$. We wish to evaluate $d_n^{(p)}=d_n^{(p)}[g_0]$. Since $c_n=\sum_{k=1}^{p}\zeta_k z_k^{-(n+1)}$ $(n=0, 1, \cdots)$, we have

$$d_n^{(p)}=\sum\begin{vmatrix} \zeta_{k_1}z_{k_1}^{-(n+2)} & \zeta_{k_2}z_{k_2}^{-(n+3)} & \cdots & \zeta_{k_p}z_{k_p}^{-(n+p+1)} \\ \zeta_{k_1}z_{k_1}^{-(n+3)} & \zeta_{k_2}z_{k_2}^{-(n+4)} & \cdots & \zeta_{k_p}z_{k_p}^{-(n+p+2)} \\ & & \cdots & \\ \zeta_{k_1}z_{k_1}^{-(n+p+1)} & \zeta_{k_2}z_{k_2}^{-(n+p+2)} & \cdots & \zeta_{k_p}z_{k_p}^{-(n+2p)} \end{vmatrix},$$

where the sum is extended over all the combinations (with repetitions) $k_1, k_2, \cdots, k_p$ of the numbers $1, 2, \cdots, p$. Since the combinations with repeated elements lead to vanishing determinants we may also write

$$d_n^{(p)}=\frac{\zeta_1\zeta_2\cdots\zeta_p}{(z_1z_2\cdots z_p)^{n+2p}}\sum\begin{vmatrix} z_{k_1}^{2p-2} & z_{k_2}^{2p-3} & \cdots & z_{k_p}^{p-1} \\ z_{k_1}^{2p-3} & z_{k_2}^{2p-4} & \cdots & z_{k_p}^{p-2} \\ \cdot & \cdot & \cdots & \cdot \\ z_{k_1}^{p-1} & z_{k_2}^{p-2} & \cdots & z_{k_p}^{0} \end{vmatrix}.$$

The sum of determinants in the last formula is known to be equal to the square of the difference-product of the numbers $z_1, z_2, \cdots, z_p$; hence

$$d_n^{(p)}[g_0]=\Delta\zeta_1\zeta_2\cdots\zeta_p(z_1z_2\cdots z_p)^{-(n+2p)}, \tag{2}$$

$$\Delta=\prod_{k<k'}(z_k-z_{k'})^2, \qquad k, k'=1, 2, \cdots, p.$$

Now suppose the $2p-1$ numbers $c_{n+1}, c_{n+2}, \cdots, c_{n+2p-1}$ are successive Taylor coefficients of the more general rational function

$$g(z)=\sum_{n=0}^{\infty}c_nz^n$$

$$=-\sum_{k=1}^{q}\left(\frac{\zeta_k}{(z-z_k)^{p_k}}+\frac{\zeta_{k,1}}{(z-z_k)^{p_k-1}}+\cdots+\frac{\zeta_{k,p_k-1}}{z-z_k}\right), \tag{3}$$

$$z_k\neq 0,$$

which has, at most, $p=p_1+p_2+\cdots+p_q$ poles (counted according to their multiplicities). We are going to evaluate $d_n^{(p)}=d_n^{(p)}[g]$ as the

limit of determinants $d_n^{(p)}$ formed for rational functions with only simple poles.

With the numbers $z_{k,1}, z_{k,2}, \cdots, z_{k,p_k}$ $(k=1, 2, \cdots, q)$ which are assumed to be different from each other and from 0 but otherwise arbitrary, we define the polynomial

$$\pi_k(z) = \zeta_k + \zeta_{k,1}(z - z_k) + \cdots + \zeta_{k,p_k-1}(z - z_k)^{p_k-1}, \tag{4}$$

and consider, instead of the function $g(z)$, the function

$$g_1(z) = \sum_{n=0}^{\infty} c_n' z^n = -\sum_{k=1}^{q} \frac{\pi_k(z)}{(z - z_{k,1})(z - z_{k,2}) \cdots (z - z_{k,p_k})}. \tag{5}$$

$g_1(z)$ has only simple poles. Its partial fraction expansion is

$$g_1(z) = -\sum_{k,l} \frac{\zeta_{k,l}'}{z - z_{k,l}}, \qquad l = 1, 2, \cdots, p_k;\ k = 1, 2, \cdots, q,$$

where

$$\zeta_{k,l}' = \frac{\pi_k(z_{k,l})}{(z_{k,l} - z_{k,1}) \cdots (z_{k,l} - z_{k,l-1})(z_{k,l} - z_{k,l+1}) \cdots (z_{k,l} - z_{k,p_k})}.$$

Applying the first result (see formula (2)) to the function $g_1(z)$, we find $d_n^{(p)}[g_1] = \Delta' \prod_{k,l} \zeta_{k,l}' z_{k,l}^{-(n+2p)}$, where Δ' is the square of the difference-product of the numbers $z_{k,l}$. But $\prod_{k,l} \zeta_{k,l}' = \prod_{k,l} \pi_k(z_{k,l}) / \prod_k \Delta_k$, where $\Delta_k = (-1)^{p_k(p_k-1)/2} \prod_{l<l'} (z_{k,l} - z_{k,l'})^2$. Hence

$$\begin{aligned} d_n^{(p)}[g_1] &= \Delta'' \prod_{k,l} \pi_k(z_{k,l}) z_{k,l}^{-(n+2p)}, \\ \Delta'' &= (-1)^{\sum_k p_k(p_k-1)/2} \prod_{k<k';\, l,l'} (z_{k,l} - z_{k',l'})^2, \\ & \qquad l = 1, 2, \cdots, p_k;\ l' = 1, 2, \cdots, p_{k'}. \end{aligned} \tag{6}$$

We now assume that the numbers $z_{k,l}$ depend on a variable δ, and that $\lim_{\delta \to 0} z_{k,l}(\delta) = z_k$ $(l=1, 2, \cdots, p_k;\ k=1, 2, \cdots, q)$. Instead of $g_1(z)$ and c_n' we now have $g(z, \delta)$ and $c_n(\delta)$ respectively. Comparing (5) and (3), we see that $g(z, \delta)$ converges to $g(z)$, $c_n(\delta)$ converges to c_n, as $\delta \to 0$. Hence $d_n^{(p)}[g(\delta)]$ converges to $d_n^{(p)}[g]$, as $\delta \to 0$. Moreover, by (4), $\pi_k(z_{k,l})$ converges to ζ_k, as $\delta \to 0$. Using all these limits, we obtain from (6)

$$\begin{aligned} d_n^{(p)}[g] &= \Delta \zeta_1^{p_1} \zeta_2^{p_2} \cdots \zeta_q^{p_q} (z_1^{p_1} z_2^{p_2} \cdots z_q^{p_q})^{-(n+2p)}, \\ \Delta &= (-1)^{\sum_k p_k(p_k-1)/2} \prod_{k<k'} (z_k - z_{k'})^{2p_k p_{k'}}, \quad k, k' = 1, 2, \cdots, q. \end{aligned} \tag{7}$$

3. **The equation for all the poles on the circle of convergence.** In the following, the numbers a_n will represent the Taylor coefficients of any analytic function that has no singularities other than poles on its circle of convergence about 0. And besides such determinants as defined in (1), we shall consider determinants $d_{ni}^{(m)}$ $(i=1, 2, \cdots, m)$, defined as

$$(8)\quad d_{ni}^{(m)}[f] = \begin{vmatrix} a_{n+1} & a_{n+2} & \cdots & a_{n+i-1} & a_n & a_{n+i+1} & \cdots & a_{n+m} \\ a_{n+2} & a_{n+3} & \cdots & a_{n+i} & a_{n+1} & a_{n+i+2} & \cdots & a_{n+m+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n+m} & a_{n+m+1} & \cdots & a_{n+i+m-2} & a_{n+m-1} & a_{n+i+m} & \cdots & a_{n+2m-1} \end{vmatrix},$$

$$d_{n-1}^{(m)}[f] = -d_{nm}^{(m)}[f], \qquad n = 0, 1, \cdots.$$

THEOREM 1. *Suppose* $f(z)=\sum_n a_n z^n = g(z)+f_1(z)$, *where*

$$g(z) = -\sum_{k=1}^{q}\left(\frac{\zeta_k}{(z-z_k)^{p_k}} + \frac{\zeta_{k,1}}{(z-z_k)^{p_k-1}} + \cdots + \frac{\zeta_{k,p_k-1}}{z-z_k}\right),$$

$$|z_k| = r > 0, \quad z_k \neq z_{k'}(k \neq k'), \quad \zeta_k \neq 0, \quad k, k' = 1, 2, \cdots, q,$$

and $f_1(z)$ *is regular in* $|z| \leqq R = \tau r$, $\tau > 1$; *suppose*

$$\pi(z) = (z-z_1)^{p_1}(z-z_2)^{p_2}\cdots(z-z_q)^{p_q}$$
$$= z^p + \gamma_1 z^{p-1} + \cdots + \gamma_p, \qquad p = p_1 + p_2 + \cdots + p_q.$$

Then, for $i=1, 2, \cdots, p$,

$$(9)\qquad \gamma_i + d_{ni}^{(p)}[f]/d_n^{(p)}[f] = O(\tau^{-n}), \qquad \text{as } n \to \infty.$$

PROOF. The function $\phi(z)=\sum_\nu \alpha_\nu z^\nu = \pi(z)f(z)$ has the coefficients

$$(10)\qquad \alpha_\nu = a_{\nu-p} + \gamma_1 a_{\nu-p+1} + \gamma_2 a_{\nu-p+2} + \cdots + \gamma_p a_\nu.$$

If, in (10), we put ν successively equal to $n+p$, $n+p+1, \cdots, n+2p-1$, we have p equations for the unknowns $\gamma_1, \gamma_2, \cdots, \gamma_p$, whose solution is, provided $d_n^{(p)}[f] \neq 0$,

$$(11)\qquad \gamma_i + d_{ni}^{(p)}/d_n^{(p)} = e_{ni}^{(p)}/d_n^{(p)}, \qquad i = 1, 2, \cdots, p,$$

where $e_{ni}^{(p)}$ denotes the determinant obtained from $d_{ni}^{(p)}$ when a_n, $a_{n+1}, \cdots, a_{n+p-1}$ are replaced by $\alpha_{n+p}, \alpha_{n+p+1}, \cdots, \alpha_{n+2p-1}$ respectively.

The determinants $e_{ni}^{(p)}$ are easily estimated. Since $\phi(z)$ is regular in $|z| \leqq R$, and $f(z)$ is regular in $|z| < r$, there is some number $\sigma > 1$

such that $\alpha_n = O(\sigma^{-n}R^{-n})$, $a_n = O(\sigma^{n/(p-1)}r^{-n})$ as $n \to \infty$; hence

$$e_{ni}^{(p)} = O(r^{-(p-1)n}R^{-n}) = O(\tau^{-n}r^{-pn}), \qquad \text{as } n \to \infty. \tag{12}$$

A lower bound for the determinants $d_n^{(p)}[f]$ is derived from identity (7). If $g(z) = \sum_n c_n z^n$, $f_1(z) = \sum_n b_n z^n$, then $a_n = b_n + c_n$, and $d_n^{(p)}[f]$ is the sum of $d_n^{(p)}[g]$ and several other determinants, each of which has at least one column of b's and otherwise columns of c's. Hence, as in (12),

$$d_n^{(p)}[f] = d_n^{(p)}[g] + O(\tau^{-n}r^{-pn}), \qquad \text{as } n \to \infty. \tag{13}$$

But since, by (7), for $n = 0, 1, 2, \cdots$, $|d_n^{(p)}[g]| = Mr^{-pn}$, where M is a positive constant, we deduce from (13) that there is another constant M_1 such that, for sufficiently large n,

$$|d_n^{(p)}[f]| > M_1 r^{-pn}. \tag{14}$$

In particular, $d_n^{(p)}[f] \neq 0$ when n is sufficiently large.

Inequalities (12) and (14), together with equation (11), prove the theorem.

Let it be noted that, since $\gamma_p = (-1)^p z_1^{p_1} z_2^{p_2} \cdots z_q^{p_q}$, we have $r^p = |\gamma_p|$, or by (9), $r^p = \lim_{n\to\infty} |d_{n-1}^{(p)}[f]/d_n^{(p)}[f]|$. This relation implies $r^p = \lim_{n\to\infty} |d_n^{(p)}[f]|^{-1/n}$.[2]

It is not necessary that all the poles z_k should lie on the circle $|z| = r$. Indeed, we may state the following:

COROLLARY. *Estimate* (9) *still holds true if* $|z_k| = \tau_k r$, $\tau_k \geq 1$ $(k = 1, 2, \cdots, q)$, *and* $R = \tau r \cdot \tau_1^{p_1}\tau_2^{p_2} \cdots \tau_q^{p_q}$, $\tau > 1$.

The proof of this corollary is an obvious variation of the proof given above.

If $f(z)$ is a rational function, say $f(z) = \psi(z)/\pi(z)$, where $\psi(z)$ is a polynomial whose degree we define to be $s \geq p-2$, then $\phi(z) = \sum_\nu \alpha_\nu z^\nu = \pi(z)f(z) = \psi(z)$; hence $\alpha_\nu = 0$ for $\nu > s$, and consequently $e_{ni}^{(p)} = 0$ for $n > s-p$. On the other hand, $f(z)$ is the sum of a polynomial $f_1(z)$ of degree $s-p$ and a fraction $g(z)$ such as in (3). Hence, for $n > s-p$, $d_n^{(p)}[f] = d_n^{(p)}[g] \neq 0$, by (7). Thus, relation (9) holds, for $n > s-p$, with 0 substituted for the second term. In particular,

[2] This "rather delicate limit relation" (see P. Dienes, *The Taylor series*, Oxford, 1931, p. 330) was first proved by Hadamard, op. cit. Other authors have tried to give simpler proofs for it, see, for example, A. Ostrowski, *Über einen Satz von Herrn Hadamard*, Jber. Deutscher Math. Verein. vol. 35 (1926) pp. 179–182. It is an immediate consequence of identity (7). On the other hand, it is the only intricate point in the proof of Theorem 1.

$$\gamma_i = -\, d^{(p)}_{s-p+1,i}[f]/d^{(p)}_{s-p+1}[f], \qquad i = 1, 2, \cdots, p. \tag{15}$$

The numbers γ_i having been found, the numbers α_ν are given by (10). Thus, the coefficients of the polynomials $\pi(z)$ and $\psi(z)$ are expressed, by (15) and (10), as rational functions of the Taylor coefficients $a_0, a_1, \cdots, a_{p+s}$ of $f(z)=\psi(z)/\pi(z)$.

4. **An inequality for persymmetric determinants.** In the following as in the foregoing we shall assume the function $f(z)=\sum_n a_n z^n$ to have no singularities other than poles on the circle of convergence $|z|=r$ of the Taylor series. We shall be interested no longer in all the poles, but only in the poles of highest order on $|z|=r$. We shall say that $f(z)$ is of order m on $|z|=r$ if $f(z)$ has there at least one pole of order m, and no other singularities but poles of order less than or equal to m.

We are going to establish an estimate for the determinants $d^{(q)}_n[f]$ $(n=0, 1, \cdots)$ where q is the number of poles of highest order of $f(z)$ on $|z|=r$. Suppose that $f(z)=\sum_n a_n z^n = g(z)+f_1(z)$, where $g(z)$ $=-\sum_{k=1}^{q}\zeta_k/(z-z_k)^m$, $|z_k|=r>0$, and $f_1(z)$ is regular in $|z|<r$ and of order less than m on $|z|=r$. The coefficient of z^n in the Taylor series expansion of $-\zeta_k(z-z_k)^{-m}$ is

$$(-1)^{m-1}\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!z_k^{m+n}}\zeta_k$$

$$= (-1)^{m-1}\frac{n^{m-1}}{(m-1)!z_k^{m+n}}\zeta_k + O\left(\frac{n^{m-2}}{r^n}\right), \qquad \text{as } n\to\infty.$$

The coefficient of z^n in the Taylor series expansion of $f_1(z)$ is, for the same reason, $O(n^{m-2}r^{-n})$. Hence we have the estimate

$$a_n = \sum_{k=1}^{q}\frac{\xi_k}{z_k^{n+1}} + O\left(\frac{n^{m-2}}{r^n}\right), \qquad \text{as } n\to\infty, \tag{16}$$

$$\xi_k = (-1)^{m-1}\frac{n^{m-1}}{(m-1)!z_k^{m-1}}\zeta_k.$$

It implies the less exact estimate

$$a_n = O(n^{m-1}/r^n), \qquad \text{as } n\to\infty. \tag{17}$$

Corresponding to the decomposition (16), the determinant $d^{(q)}_n[f]$ splits into the sum of a persymmetric determinant whose terms are $c_\nu=\sum_{k=1}^{q}\xi_k z_k^{-(\nu+1)}$ $(\nu=n+1, n+2, \cdots, n+2q-1)$ and several other determinants, each of which has at least one column made of terms that are $O(n^{m-2}r^{-n})$, and otherwise columns made of terms that are

$O(n^{m-1}r^{-n})$. The first determinant, as was shown in §2, is equal to $\Delta\cdot\xi_1\xi_2\cdots\xi_q\,(z_1z_2\cdots z_q)^{-(n+2q)}$, where Δ is the square of the difference-product of the numbers $z_1, z_2, \cdots, z_q$. The sum of the other determinants is $O(n^{q(m-1)-1}r^{-qn})$. Hence, using (16), we have the estimate

$$d_n^{(q)}[f] = C_f\frac{n^{q(m-1)}}{(z_1z_2\cdots z_q)^n} + O\left(\frac{n^{q(m-1)-1}}{r^{qn}}\right), \text{ as } n\to\infty, \tag{18}$$

$$C_f = (-1)^{q(m-1)}\Delta\cdot\frac{\zeta_1\zeta_2\cdots\zeta_q}{[(m-1)!]^q(z_1z_2\cdots z_q)^{2q+m-1}}.$$

5. The equation for the poles of highest order. In §3 we have established the algebraic equation whose roots are *all* the poles of $f(z)$ (counted according to their multiplicities) on $|z|=r$. We shall now establish the algebraic equation whose roots are the different poles of highest order of $f(z)$ on $|z|=r$.

THEOREM 2. *Suppose that* $f(z)=\sum_n a_nz^n=g(z)+f_1(z)$, *where*

$$g(z) = -(\zeta_1/(z-z_1)^m + \zeta_2/(z-z_2)^m + \cdots + \zeta_q/(z-z_q)^m),$$

$$|z_k| = r > 0, \quad z_k \neq z_{k'}(k\neq k'), \quad \zeta_k\neq 0, \quad k, k' = 1, 2, \cdots, q,$$

and $f_1(z)$ *is regular in* $|z|<r$ *and of order less than* m *on* $|z|=r$; *suppose* $\pi(z)=(z-z_1)(z-z_2)\cdots(z-z_q)=z^q+\gamma_1z^{q-1}+\cdots+\gamma_q$. *Then, for* $i=1, 2, \cdots, q$,

$$\gamma_i + d_{ni}^{(q)}[f]/d_n^{(q)}[f] = O(1/n), \qquad \text{as } n\to\infty, \tag{19a}$$

and

$$m = 1 + \lim_{n\to\infty}\frac{\log|d_n^{(q)}[f]| + nq\log r}{q\log n}. \tag{19b}$$[3]

PROOF. The function $\phi(z)=\sum_\nu\alpha_\nu z^\nu=\pi(z)f(z)$, whose coefficients are

$$\alpha_\nu = a_{\nu-q} + \gamma_1a_{\nu-q+1} + \cdots + \gamma_qa_\nu, \tag{20}$$

is regular in $|z|<r$, and of order less than or equal to $m-1$ on $|z|=r$. Hence, as in (17), the coefficients α_ν are $O(\nu^{m-2}r^{-\nu})$, as $\nu\to\infty$. If, in

[3] Hadamard, see op. cit., defined the order of any series $\sum a_nz^n$ and proved that, on its circle of convergence, the order is given by the formula $m=1+\limsup(\log|a_n|+n\log r)/\log n$. Formula (19b) represents, in accordance with the objective of this paper, the order as a regular limit, but it holds only for series which are meromorphic on their circle of convergence.

(20), we put ν successively equal to $n+q$, $n+q+1$, $\cdots$, $n+2q-1$, we have q equations for the unknowns $\gamma_1, \gamma_2, \cdots, \gamma_q$, whose solution is, provided $d_n^{(q)}[f] \neq 0$,

$$\gamma_i + d_{ni}^{(q)}/d_n^{(q)} = e_{ni}^{(q)}/d_n^{(q)}, \qquad i = 1, 2, \cdots, q, \tag{21}$$

where $e_{ni}^{(q)}$ denotes the determinant that is obtained from $d_{ni}^{(q)}$ when $a_n, a_{n+1}, \cdots, a_{n+q-1}$ are replaced by $\alpha_{n+q}, \alpha_{n+q+1}, \cdots, \alpha_{n+2q-1}$ respectively. Using estimate (17) for the a's and a corresponding estimate for the α's, we find readily

$$e_{ni}^{(q)} = O(n^{q(m-1)-1} r^{-qn}), \qquad \text{as } n \to \infty. \tag{22}$$

Taking into account that, by assumption, the poles $z_1, z_2, \cdots, z_q$ are different from each other, we deduce from relation (18) that there are constants M_1, M_2 such that, for sufficiently large n,

$$d_n^{(q)}[f] = \mu_n n^{q(m-1)} r^{-qn}, \qquad M_1 < |\mu_n| < M_2. \tag{23}$$

In particular, $d_n^{(q)}[f] \neq 0$ when n is sufficiently large.

Inequalities (22) and (23), together with equation (21), prove the theorem.

6. **Converses of Theorems 1 and 2.** In problems where the poles of $f(z)$ are to be determined from the given Taylor coefficients, it is, in general, not known, a priori, what and how many singularities $f(z)$ has on the circle of convergence $|z| = r$ of its Taylor series expansion. Even if $f(z)$ is known to be meromorphic, Theorems 1 and 2 do not always suffice to establish the polynomial $\pi(z)$ whose zeros are all the poles, or the poles of highest order, of $f(z)$ on $|z| = r$, as the numbers p or q may not be known. In such a case the procedure to follow is to investigate the behavior of the quotients $d_{ni}^{m}[f]/d_n^{(m)}[f]$, as $n \to \infty$, successively for $m = 1, 2, \cdots$. From the convergence properties of these sequences the nature and number of the singularities of $f(z)$ on $|z| = r$ can be deduced, as will be shown in the following converses of Theorems 1 and 2. It will always be assumed that $f(z) = \sum_n a_n z^n$ has a positive radius of convergence.

THEOREM 3. *Suppose that, for a fixed p, there are p numbers $\gamma_1, \gamma_2, \cdots, \gamma_p$ and a number $\tau > 1$ such that, for $i = 1, 2, \cdots, p$,*

$$\gamma_i + d_{ni}^{(p)}[f]/d_n^{(p)}[f] = O(\tau^{-n}), \qquad \text{as } n \to \infty; \tag{24}$$

suppose that this is not the case for any $p' < p$. Then $f(z)$ is regular in

the circle $|z| \leqq r = |\gamma_p|^{1/p}$ *but for* p *poles (counted according to their multiplicities) on* $|z| = r$, *which are the zeros of the polynomials* $\pi(z) = z^p + \sum_{i=1}^{p} \gamma_i z^{p-i}$.

PROOF. According to the definition of the determinants $d_{ni}^{(p)}$, $d_n^{(p)}$, and since the quotients $d_{ni}^{(p)}/d_n^{(p)}$ exist, by assumption, for sufficiently large n, we have

$$(25) \quad a_n - (1/d_n^{(p)})(a_{n+1}d_{n1}^{(p)} + a_{n+2}d_{n2}^{(p)} + \cdots + a_{n+p}d_{np}^{(p)}) = 0.$$

Let r be the radius of convergence of $\sum_n a_n z^n$; then $a_n = O(r'^{-n})$, as $n \to \infty$, for any $r' < r$. If we choose r' so that $R = rr' > r$, we obtain, by substituting (24) in (25),

$$\alpha_{n+p} = a_n + \gamma_1 a_{n+1} + \gamma_2 a_{n+2} + \cdots + \gamma_p a_{n+p} = O(R^{-n}), \text{ as } n \to \infty.$$

Hence the function $\phi(z) = \pi(z)f(z) = \sum_n \alpha_n z^n$ is regular in $|z| \leqq R > r$. This implies that $f(z)$ has no other singularities than, at most, p poles on the circle $|z| = r$, which are among the zeros of $\pi(z)$.

If $f(z)$ had less than p poles on $|z| = r$, some relation like (24) would, by Theorem 1, hold for $p' < p$, in contradiction with the hypothesis. Hence $f(z)$ has exactly p poles on $|z| = r$, which are the zeros of $\pi(z)$. Since $(-1)^p \gamma_p$ is the product of the p zeros, $r = |\gamma_p|^{1/p}$.

THEOREM 4. *Suppose* $f(z)$ *has no singularities but poles on* $|z| = r$; *suppose that, for a fixed* q, *there are* q *numbers* $\gamma_1, \gamma_2, \cdots, \gamma_q$ *such that, for* $i = 1, 2, \cdots, q$,

$$(26) \quad \gamma_i + \lim_{n \to \infty} d_{ni}^{(q)}[f]/d_n^{(q)}[f] = 0;$$

suppose that this is not the case for any $q' < q$. *Then* $f(z)$ *is regular in the circle* $|z| < r = |\gamma_q|^{1/q}$, *and of order*

$$m = 1 + \lim_{n \to \infty} (\log |d_n^{(q)}| + nq \log r)/q \log n$$

on $|z| = r$. $f(z)$ *has* q *different poles of order* m *on* $|z| = r$, *which are the zeros of the polynomial* $\pi(z) = z^q + \sum_{i=1}^{q} \gamma_i z^{q-i}$.

PROOF. Let r be the radius of convergence of $\sum_n a_n z^n$, and let m be the order of $f(z)$ on $|z| = r$. If $f(z)$ had less than q poles on $|z| = r$, some relation like (26) would, by Theorem 2, hold true for $q' < q$, in contradiction with the hypothesis. Hence $f(z)$ has $q' \geqq q$ poles of order m on $|z| = r$.

As in the proof of Theorem 3, we have, for sufficiently large n,

$$(27) \quad a_n - (1/d_n^{(q)})(a_{n+1}d_{n1}^{(q)} + a_{n+2}d_{n2}^{(q)} + \cdots + a_{n+q}d_{nq}^{(q)}) = 0.$$

In §4 (see (17)) it was shown that $a_n = O(n^{m-1}r^{-n})$, as $n \to \infty$, whereas by assumption, $d_{n1}^{(q)}/d_n^{(q)} = -\gamma_i + o(1)$, as $n \to \infty$. Hence (27) implies

$$(28) \qquad \alpha_{n+q} = a_n + \gamma_1 a_{n+1} + \gamma_2 a_{n+2} + \cdots + \gamma_q a_{n+q} = o(n^{m-1}r^{-n}),$$

as $n \to \infty$.

Now suppose $q' > q$. Since $\pi(z)$ is of degree q, $\phi(z) = \sum_n \alpha_n z^n = \pi(z)f(z)$ is, like $f(z)$, of order m, and has $q'' \geqq q' - q$ different poles $z_1, z_2, \cdots, z_{q''}$ of order m on $|z| = r$. Hence, as was shown in §4 (see (16)) α_n has the form

$$(29) \qquad \alpha_n = n^{m-1} \sum_{k=1}^{q''} \frac{\theta_k}{z_k^n} + O\left(\frac{n^{m-2}}{r^n}\right)$$

as $n \to \infty$, and $\theta_k \neq 0$ $(k = 1, 2, \cdots, q'')$.

It is easily shown[4] that there is a positive number M such that, for $n = 0, 1, \cdots$, at least one of the q'' numbers $\left| r^{n+\nu} \sum_{k=1}^{q''} \theta_k / z_k^{n+\nu} \right|$, $\nu = 0, 1, \cdots, q'' - 1$ is greater than M. This fact, together with (29), implies that, to any given n_0, there are numbers $n > n_0$ such that $|\alpha_n| > (1/2)Mn^{m-1}r^{-n}$, whereas, because of (28), n_0 may be found such that, for *all* $n > n_0$, $|\alpha_n| < (1/2)Mn^{m-1}r^{-n}$. Thus the assumption $q' > q$ is proved to be wrong.

Since we have proved that the number of poles of $f(z)$ of order m on $|z| = r$ is q, the other statements of the theorem follow immediately from Theorem 2.

COROLLARY. *If* $f(z) = \sum_n a_n z^n$ *has no singularities but poles on the circle of convergence* $|z| = r$ *of* $\sum_n a_n z^n$, *and if* $z_1 = \lim_{n\to\infty} a_n/a_{n+1}$ *exists, then* $f(z)$ *is of order* $m = 1 + \lim_{n\to\infty} (\log |a_n| + n \log r)/\log n$ *on* $|z| = r$, z_1 *being the only pole of order* m *on* $|z| = r$.[5]

This is obviously Theorem 4 for $q = 1$.

The following theorem is the logical product of Theorems 3 and 4.

THEOREM 5. *Suppose that, for a fixed* p, *there are* p *numbers* $\gamma_1, \gamma_2, \cdots, \gamma_p$ *and a number* $\tau > 1$ *such that, for* $i = 1, 2, \cdots, p$, $\gamma_i + d_{n1}^{(p)}[f]/d_n^{(p)}[f] = O(\tau^{-n})$, *as* $n \to \infty$; *suppose that there is no* $p' < p$ *such that the limits* $\lim_{n\to\infty} d_{n1}^{(p')}[f]/d_n^{(p')}[f]$ $(i = 1, 2, \cdots, p')$ *exist. Then* $f(z)$ *is regular in the circle* $|z| \leqq r = |\gamma_p|^{1/p}$ *but for* p *poles of the first order on* $|z| = r$, *which are the zeros of the polynomial* $\pi(z) = z^p + \sum_{i=1}^{p} \gamma_i z^{p-i}$.

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[4] See A. Ostrowski, *Über Singularitäten gewisser mit Lücken behafteten Potenzreihen*, Jber. Deutschen Math. Verein. vol. 35 (1926) p. 269.

[5] Cf. footnote 3.

PROOF. As in the proof of Theorem 3 it is seen that $f(z)$ has no other singularities than, at most, p poles (counted according to their multiplicities) on the circle of convergence $|z|=r$ of $f(z)=\sum_n a_n z^n$. If, on $|z|=r$, $f(z)$ had less than p poles, or if some of the poles were of higher than the first order, then $f(z)$ would have $q<p$ poles of highest order on $|z|=r$. This would imply, by Theorem 2, the existence of the limits $\lim_{n\to\infty} d_{ni}^{(q)}/d_n^{(q)}$ $(i=1, 2, \cdots, q)$, in contradiction with the hypothesis of the theorem. $f(z)$ has, therefore, exactly p poles of the first order on $|z|=r$. The other statements of the theorem follow immediately.

7. **Application to the evaluation of roots.** The zeros of smallest absolute value of the function $F(z)=\sum_n A_n z^n$ are the poles of the function $f(z)=\sum_n a_n z^n=1/F(z)$ on its circle of convergence. Hence the theory just developed contains a method of evaluating zeros of smallest absolute value of functions given by Taylor series. The procedure to follow is that outlined at the beginning of §6. It requires the evaluation of the determinants $d_{ni}^{(m)}[f]$ $(n=0, 1, \cdots)$ for several values of m. These determinants are defined, in (8), in terms of the coefficients a_n of $f(z)$. They also can be expressed[6] easily in terms of the coefficients A_n of the given function $F(z)$, with the determinant $d_{ni}^{(m)}[f]$ of order m in the elements $a_n, a_{n+1}, \cdots, a_{n+2m-1}$ replaced by a determinant of order $m+n+1$ in the elements $A_0, A_1, \cdots, A_{n+2m-1}$.

Since the order of these latter determinants increases with n, they are not suited for practical purposes. If numerical evaluation of the zeros of $F(z)$ is required, it is advisable to compute first the coefficients a_n of the reciprocal function $f(z)$, and then to work with the determinants $d_{ni}^{(m)}[f]$ as given in (8). If $f(z)=\sum_n a_n z^n$ is the negative reciprocal of $F(z)=\sum_n A_n z^n$, and the coefficient A_0 is made to be -1, then the coefficients a_n are given by the simple recursion formulae:

$$
\begin{aligned}
a_0 &= 1, \\
a_1 &= A_1, \\
a_2 &= A_2 + a_1 A_1, \\
a_3 &= A_3 + a_1 A_2 + a_2 A_1, \\
&\cdots\cdots\cdots\cdots.
\end{aligned}
$$

To illustrate the numerical efficiency of the method, the two conjugate complex roots z_1, z_2 of smallest absolute value of $F(z)=z-e^z=0$ were computed. The results are exhibited in the following table. The

[6] See J. Hadamard, loc. cit., p. 136.

second column of the table contains the Taylor coefficients a_n of $f(z)=1/(e^z-z)$ multiplied by $n!$ in order to avoid fractions; the third column contains the quotients $d_{n1}^{(2)}[f]/d_n^{(2)}[f]$, which represent successive approximations to $-\gamma_1=z_1+z_2$; the fourth column contains the quotients $d_{n-1}^{(2)}[f]/d_n^{(2)}[f]$, which represent successive approximations to $\gamma_2=z_1z_2$. The final values in the table give

$$z_1+z_2=0.6362630104, \qquad z_1z_2=1.889406976$$

with a relative error of $1\cdot10^{-10}$. As the table shows, the fifth approximation, which is obtained with very little effort, already gives the values of z_1+z_2 and z_1z_2 with a relative error of $1\cdot10^{-6}$.

n	a_n	$d_{n1}^{(2)}[f]/d_n^{(2)}[f]$	$d_{n-1}^{(2)}[f]/d_n^{(2)}[f]$
1	0	0.67	2
2	−1	0.632	1.89
3	−1	0.6368	1.891
4	5	0.63616	1.8897
5	19	0.6362624	1.88942
6	−41	0.6362628	1.88941
7	−519	0.63626291	1.8894073
8	−183	0.63626306	1.8894072
9	19 223	0.63626299	1.88940700
10	73 451	0.636263011	1.889406978
11	−847 067	0.6362630099	1.889406977
12	−8 554 547	0.6362630105	1.889406976
13	32 488 611	0.6362630104	1.889406976
14	977 198 559	0.6362630104	1.889406976
15	1 325 135 969		
16	−116 987 762 287		

PURDUE UNIVERSITY

ON EXTENSION OF WRONSKIAN MATRICES

DAVID M. KRABILL

1. **Introduction.** By an interval J we shall understand a finite interval of the type $a \leqq x \leqq b$. If $u_1(x), u_2(x), \cdots, u_n(x)$ are real functions possessing finite derivatives of the first t orders in an interval J and $0 \leqq s \leqq t$, we call the functional matrix

$$M_s(u_1, \cdots, u_n) \equiv \begin{Vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdots & \cdot \\ u_1^{(s)} & u_2^{(s)} & \cdots & u_n^{(s)} \end{Vmatrix}$$

their Wronskian matrix of order s. The Wronskian $W(u_1, \cdots, u_n)$ is the determinant of the matrix $M_{n-1}(u_1, \cdots, u_n)$.

The principal result we obtain is that if $n \leqq s \leqq t$ and the Wronskian matrix of order s for n arbitrary functions $u_1(x), u_2(x), \cdots, u_n(x)$ of class[1] $C^{(t)}$ in an interval J has constant rank n, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that the extended matrix $M_s(u_1, \cdots, u_{n+1})$ has constant rank $n+1$ in J. We employ a theorem of Curtiss[2] which may be stated in the form:

THEOREM C. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ in an interval J and their Wronskian matrix of order t has rank n throughout J, then the Wronskian $W(u_1, \cdots, u_n)$ has at most isolated zeros.*

From the extension property of Wronskian matrices we obtain a sufficient condition, in terms of the rank of a certain functional matrix, that an arbitrary set of functions having suitable class properties be solutions of an ordinary homogeneous linear differential equation.

2. **Lemmas.** We first prove two lemmas.

LEMMA 1. *If $\delta, c_1, c_2, \cdots, c_n$ are given constants with $\delta > 0$, there exists a function $f(x)$ of class $C^{(n)}$ in the interval $-1 \leqq x \leqq 1$ which satisfies the conditions*: (1) $|f(x)| \leqq \delta$, $-1 \leqq x \leqq 1$; (2) $f^{(i)}(-1) = f^{(i)}(1) = 0$, $i = 0, 1, \cdots, n$; (3) $f(0) = 0$, $f^{(i)}(0) = c_i$, $i = 1, 2, \cdots, n$.

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[1] Each function has continuous derivatives of the first t orders at every point of J.

[2] D. R. Curtiss, *The vanishing of the Wronskian and the problem of linear dependence*, Math. Ann. vol. 65 (1908) Theorem 4.

PROOF. Let $p(x)$ be the polynomial

$$c_1x + (c_2/2!)x^2 + \cdots + (c_n/n!)x^n.$$

Then $p(0)=0$, $p^{(i)}(0)=c_i$ and for some β with $0<\beta<1$, $|p(x)| \leqq \delta$, $-\beta \leqq x \leqq \beta$. Take α such that $0<\alpha<\beta$. Let

$$\theta = (-1)^n(x+\alpha)^{n+1}/(x+\beta)^2, \qquad \phi = -(x-\alpha)^{n+1}/(x-\beta)^2,$$

and let

$$g_1(x) \equiv \begin{cases} 0, & x \leqq -\beta, \\ e^\theta, & -\beta < x < -\alpha, \\ 1, & -\alpha \leqq x \leqq 0, \end{cases} \qquad g_2(x) \equiv \begin{cases} 1, & 0 \leqq x \leqq \alpha, \\ e^\phi, & \alpha < x < \beta, \\ 0, & \beta \leqq x. \end{cases}$$

The g's are of class $C^{(n)}$ in their respective intervals of definition and $g_1^{(i)}(0)=g_2^{(i)}(0)$ for $i=0, 1, \cdots, n$. Then the function

$$f(x) \equiv \begin{cases} p(x)g_1(x), & -1 \leqq x \leqq 0, \\ p(x)g_2(x), & 0 \leqq x \leqq 1, \end{cases}$$

is of class $C^{(n)}$ in the interval $-1 \leqq x \leqq 1$ and satisfies the prescribed conditions.

LEMMA 2. *Given an interval J and constants a_i and b_i, $i=0, 1, \cdots, n$ with $a_0b_0>0$, there exists a function $f(x)$ which satisfies the conditions*: (1) *$f(x)$ is nonzero and of class $C^{(n)}$ in J*; (2) $f^{(i)}(a)=a_i$, $f^{(i)}(b)=b_i$, $i=0, 1, \cdots, n$.

PROOF. Let $\delta = 1/3 \min(|a_0|, |b_0|, b-a)$. Take $f_k(x)$, $k=1, 2$, of class $C^{(n)}$ in the interval $-1 \leqq x \leqq 1$ such that: $|f_k(x)| \leqq \delta$, $-1 \leqq x \leqq 1$; $f_k^{(i)}(-1)=f_k^{(i)}(1)=f_k(0)=0$, $i=0, 1, \cdots, n$; $f_1^{(i)}(0)=a_i$, $f_2^{(i)}(0)=b_i$, $i=1, 2, \cdots, n$. Let

$$u(x) \equiv f_1((x-a)/\delta) + a_0, \qquad a \leqq x \leqq a+\delta,$$
$$v(x) \equiv f_2((x-b)/\delta) + b_0, \qquad b-\delta \leqq x \leqq b.$$

Then $u(a)=u(a+\delta)=a_0$; $|u(x)-u(a)| \leqq \delta$, $a \leqq x \leqq a+\delta$; $u^{(i)}(a)=a_i$, $u^{(i)}(a+\delta)=0$, $i=1, 2, \cdots, n$; and $v(b)=v(b-\delta)=b_0$; $|v(x)-v(b)| \leqq \delta$, $b-\delta \leqq x \leqq b$; $v^{(i)}(b)=b_i$, $v^{(i)}(b-\delta)=0$, $i=1, 2, \cdots, n$. Moreover, both $u(x)$ and $v(x)$ are nonzero in their respective mutually exclusive intervals of definition by our choice of δ. Let $w(x) \equiv (a_0-b_0)e^\theta+b_0$, $a+\delta \leqq x \leqq b-\delta$, where $\theta = -(x-a-\delta)^{n+1}/(x-b+\delta)^2$. Then $w(x)$ is nonzero throughout its interval of definition and

$$w(a+\delta)=a_0,\ w(b-\delta)=b_0,\ w^{(i)}(a+\delta)=w^{(i)}(b-\delta)=0, \quad i=1, 2, \cdots, n.$$

Therefore the function

$$f(x) \equiv \begin{cases} u(x), & a \leqq x \leqq a+\delta, \\ w(x), & a+\delta \leqq x \leqq b-\delta, \\ v(x), & b-\delta \leqq x \leqq b, \end{cases}$$

satisfies the conditions of the lemma.

3. **Extension of Wronskian matrices.** Let $u(x)$ be a function of class $C^{(t)}$, $t \geqq 1$, in an interval J with $M_1(u)$ of rank 1 at every point of J. If $v(x)$ of class $C^{(t)}$ exists such that $M_1(u, v)$ has rank 2 throughout J, the Wronskian $W(u, v)$ is nonzero in J. Thus in this case there exists a function $K(x)$ of class $C^{(t-1)}$ and nonzero in J such that $v(x)$ is a solution of the differential equation

$$u(x)y' - u'(x)y = K(x). \tag{1}$$

This leads us to ask: does (1) have a solution of class $C^{(t)}$ in J for arbitrary $K(x)$ of class $C^{(t-1)}$ which is nonzero in J? We give a counter-example.

Example. Let J be the interval $0 \leqq x \leqq 2$ and take $u(x) = 1-x$, $K(x) = 1 + |1-x|^{1/2}$. Then the differential equation (1) becomes

$$(1-x)y' + y = 1 + |1-x|^{1/2}, \tag{2}$$

for which the general solution in the interval $0 \leqq x < 1$ is

$$y(x) = u(x) \int_0^x (K(z)/u^2(z))dz + cu(x)$$

where c is constant. Thus

$$y'(x) = (1/u(x))[K(x) + u'(x)y(x)] \equiv 3 - c - (1-x)^{-1/2}$$

and $\lim_{x \to 1-0} y'(x)$ does not exist, so that the differential equation (2) has no solution of class C' defined in the interval $0 \leqq x \leqq 2$.

We now consider the special case of extending a Wronskian matrix of n columns and $n+1$ rows which has rank n throughout an interval J.

THEOREM 1. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ $(t \geqq n)$ in an interval J and their Wronskian matrix of order n has rank n at every point of J, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that $M_n(u_1, \cdots, u_{n+1})$ has rank $n+1$ throughout J.*

PROOF. Since $M_n(u_1, \cdots, u_n)$ has rank n throughout J, it follows

from Theorem C that $W(u_1, \cdots, u_n)$ has at most a finite number of zeros in J. Denote by

$$f_r(x), \qquad r = 0, 1, \cdots, n, \tag{3}$$

the cofactor of $y^{(r)}$ in the determinant $W(u_1, \cdots, u_n, y)$ in which y is unknown. If we prove the existence of a function $K(x)$ which is nonzero and of class $C^{(i-n)}$ in J for which the differential equation

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = K(x) \tag{4}$$

has a solution defined in J, we shall have proved our theorem since the coefficients in (4) are of class $C^{(i-n)}$. If $f_n(x)$ does not vanish in J, standard existence theorems apply for arbitrary $K(x)$.

Let $f_n(x)$ vanish in J. Since $f_n(x) \equiv W(u_1, \cdots, u_n)$, its zeros are finite in number. Assume that the zeros are at the points $c_1, c_2, \cdots, c_m$ with $c_1 < c_2 < \cdots < c_m$. If the c's are not all interior points of J, let

$$U_i(x) \equiv \begin{cases} \sum_{j=0}^{i} \frac{1}{j!} u_i^{(j)}(a)(x-a)^j, & x < a, \\ u_i(x), & a \leqq x \leqq b, \\ \sum_{j=0}^{i} \frac{1}{j!} u_i^{(j)}(b)(x-b)^j, & x > b, \end{cases} \qquad i = 1, 2, \cdots, n.$$

Then $W(U_1, \cdots, U_n)$ is identical with $W(u_1, \cdots, u_n)$ in J, and is a polynomial for x not in J. Therefore $W(U_1, \cdots, U_n)$ has isolated zeros and for some $a^* < a$ and $b^* > b$, the c's are its only zeros in the interval $a^* \leqq x \leqq b^*$. Consequently we assume without loss of generality that $c_1 > a$ and $c_m < b$.

Since $M_n(u_1, \cdots, u_n)$ has rank n throughout J, for each i, $i = 1, 2, \cdots, m$, at least one function in (3) does not vanish at c_i. Denote by $f_{k_i}(x)$ a function from (3) such that $f_{k_i}(c_i) \neq 0$, $i = 1, 2, \cdots, m$. Define

$$q_i(x) \equiv \pm (1/k_i!)(x - c_i)^{k_i}, \qquad i = 1, 2, \cdots, m,$$

where the ambiguous sign is so chosen that each Wronskian $W(u_1, \cdots, u_n, q_i)$, which has the value $\pm f_{k_i}(c_i)$ at c_i, has the same sign at c_i. Then, for some $\epsilon > 0$ with $a < c_1 - \epsilon < c_1 + \epsilon < c_2 - \epsilon < \cdots < c_m + \epsilon < b$,

$$Q_i(x) \equiv W(u_1, \cdots, u_n, q_i) \neq 0, c_i - \epsilon \leqq x \leqq c_i + \epsilon, i = 1, 2, \cdots, m,$$

and $q_i(x)$ is a solution of the differential equation

$$W(u_1, \cdots, u_n, y) = Q_i(x), \qquad i = 1, 2, \cdots, m.$$

Denote the nterval

$$c_{k-1}+\epsilon \leqq x \leqq c_k-\epsilon, \quad k=1, 2, \cdots, m+1, \tag{5}$$

where $c_0+\epsilon=a$ and $c_{m+1}-\epsilon=b$, by J_k. Let $P_k(x)$ be any function (Lemma 2) which is nonzero and of class $C^{(t-n)}$ in J_k and satisfies the conditions

$$\begin{aligned} P_k^{(j)}(c_{k-1}+\epsilon+0) &= Q_{k-1}^{(j)}(c_{k-1}+\epsilon-0), \\ P_k^{(j)}(c_k-\epsilon-0) &= Q_k^{(j)}(c_k-\epsilon+0), \end{aligned} \tag{6}$$

for $j=0, 1, \cdots, t-n$, where

$$\begin{aligned} Q_0^{(j)}(c_0+\epsilon-0) &= Q_1^{(j)}(c_1-\epsilon+0), \\ Q_{m+1}^{(j)}(c_{m+1}-\epsilon+0) &= Q_m^{(j)}(c_m+\epsilon-0). \end{aligned}$$

Then each function

$$K_k(x) \equiv \begin{cases} P_k(x), & c_{k-1}+\epsilon \leqq x \leqq c_k-\epsilon, \\ Q_k(x), & c_k-\epsilon \leqq x \leqq c_k+\epsilon, \end{cases} \qquad k=1, 2, \cdots, m+1,$$

where $c_{m+1}+\epsilon=b$, is nonzero and of class $C^{(t-n)}$ in its interval of definition by the second set of conditions in (6). Thus the function $K(x)$ defined in J by

$$K(x) \equiv K_k(x), \quad c_{k-1}+\epsilon \leqq x \leqq c_k+\epsilon, \quad k=1, 2, \cdots, m+1, \tag{7}$$

is nonzero and of class $C^{(t-n)}$ in J by the first set of conditions in (6).

Since $f_n(x) \neq 0$ in J_1, the differential equation $W(u_1, \cdots, u_n, y) = P_1(x)$ has a solution $p_1(x)$ which satisfies the initial conditions

$$p_1^{(j)}(c_1-\epsilon-0) = q_1^{(j)}(c_1-\epsilon+0), \qquad j=0, 1, \cdots, n-1. \tag{8}$$

It follows from the second set of conditions in (6) that (8) also holds for $j=n, n+1, \cdots, t$. The function

$$y_1(x) \equiv \begin{cases} p_1(x), & a \leqq x \leqq c_1-\epsilon, \\ q_1(x), & c_1-\epsilon \leqq x \leqq c_1+\epsilon, \end{cases}$$

is therefore a solution of the differential equation $W(u_1, \cdots, u_n, y) = K_1(x)$.

In J_2 the differential equation

$$W(u_1, \cdots, u_n, y) = P_2(x) \tag{9}$$

has a solution $p_2(x)$ with $p_2^{(j)}(c_1+\epsilon+0) = y_1^{(j)}(c_1+\epsilon-0)$, $j=0, 1, \cdots, t$,

and also a solution $r(x)$ with $r^{(j)}(c_2-\epsilon-0)=q_2^{(j)}(c_2-\epsilon+0)$, $j=0, 1, \cdots, t$. Now the difference of any two solutions of (9) is a solution of the corresponding homogeneous equation for which the u's are linearly independent solutions. Thus, for some set of constants $a_1, a_2, \cdots, a_n$,

$$p_2(x) \equiv r(x) + a_1u_1(x) + \cdots + a_nu_n(x), \quad c_1 + \epsilon \leqq x \leqq c_2 - \epsilon.$$

Then the function

$$y_2(x) \equiv \begin{cases} p_2(x), & c_1 + \epsilon \leqq x \leqq c_2 - \epsilon, \\ q_2(x) + a_1u_1(x) + \cdots + a_nu_n(x), & c_2 - \epsilon \leqq x \leqq c_2 + \epsilon, \end{cases}$$

is of class $C^{(t)}$ in its interval of definition with

$$y_2^{(j)}(c_1 + \epsilon + 0) = y_1^{(j)}(c_1 + \epsilon - 0), \qquad j = 0, 1, \cdots, t,$$

and is a solution of the differential equation $W(u_1, \cdots, u_n, y) = K_2(x)$.

We continue in the above manner, obtaining $y_k(x)$, $c_{k-1}+\epsilon \leqq x \leqq c_k + \epsilon$ $(k=2, 3, \cdots, m+1)$ of class $C^{(t)}$ in its interval of definition with

$$y_k^{(j)}(c_{k-1} + \epsilon + 0) = y_{k-1}^{(j)}(c_{k-1} + \epsilon - 0), \qquad j = 0, 1, \cdots, t,$$

which is a solution of the differential equation $W(u_1, \cdots, u_n, y) = K_k(x)$. Then the function $u_{n+1}(x)$ defined in J by

$$u_{n+1}(x) \equiv y_k(x), \quad c_{k-1} + \epsilon \leqq x \leqq c_k + \epsilon, \qquad k = 1, 2, \cdots, m+1,$$

is of class $C^{(t)}$ in J and is a solution of (4) with $K(x)$ defined by (7).

In the following theorem we consider the general case of extending a Wronskian matrix.

THEOREM 2. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ $(t \geqq n)$ in an interval J and their Wronskian matrix of order s $(n \leqq s \leqq t)$ has rank n at every point of J, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that $M_s(u_1, \cdots, u_{n+1})$ has rank $n+1$ throughout J.*

PROOF. The case $s=n$ is treated in Theorem 1. Assume now that $M_n(u_1, \cdots, u_n)$ does not have rank n throughout J. Since $M_s(u_1, \cdots, u_n)$ has rank n at every point, it follows from Theorem C that $W(u_1, \cdots, u_n)$ has isolated zeros. Define $f_r(x)$, $r=0, 1, \cdots, n$, as in (3). Since the f's are the n-rowed minor determinants of $M_n(u_1, \cdots, u_n)$ they have at least one zero in common, but only a finite number for $f_n(x) \equiv W(u_1, \cdots, u_n)$. Assume that the common zeros of the f's are at the points $c_1, c_2, \cdots, c_m$ with $c_1 < c_2 < \cdots < c_m$. There is no loss of generality to assume that $c_1 > a$ and $c_m < b$.

For each i, $i=1, 2, \cdots, m$, some n-rowed determinant of $M_s(u_1, \cdots, u_n)$, which we shall denote by Δ_i, does not vanish at c_i. If $u(x)$ is a function of class $C^{(t)}$ in J and row r of $M_s(u_1, \cdots, u_n)$ is not a row of Δ_i, denote by $\Delta_{i,r}$ the determinant of $M_s(u_1, \cdots, u_n, u)$ which consists of row r and the rows represented in Δ_i.

Let row r_i be the first row of $M_s(u_1, \cdots, u_n)$ which is not a row of Δ_i. Define

$$q_i(x) \equiv (1/(r_i-1)!)(x-c_i)^{r_i-1}, \qquad i=1, 2, \cdots, m.$$

Then at c_i the determinant Δ_{i,r_i} of $M_s(u_1, \cdots, u_n, q_i)$ has the value $(-1)^{n+1+r_i}\Delta_i(c_i)$, hence for some $\delta>0$ with $a<c_1-\delta<c_1+\delta<c_2-\delta < \cdots <c_m+\delta<b$,

$$\Delta_{i,r_i} \neq 0, \; c_i-\delta \leqq x \leqq c_i+\delta, \qquad i=1, 2, \cdots, m. \tag{10}$$

Since $M_s(u_1, \cdots, u_n, q_i)$ has rank $n+1$ throughout the interval $c_i-\delta \leqq x \leqq c_i+\delta$ by (10), $W(u_1, \cdots, u_n, q_i)$ has isolated zeros in this interval. Thus, for some $\epsilon>0$ with $\epsilon \leqq \delta$,

$$Q_i(x) \equiv W(u_1, \cdots, u_n, q_i) \neq 0 \text{ at } x=c_i \pm \epsilon, \; i=1, 2, \cdots, m. \tag{11}$$

We shall assume that $Q_{i+1}(c_{i+1}-\epsilon)$, $i=1, 2, \cdots, m-1$, is positive or negative according as $Q_i(c_i+\epsilon)$ is positive or negative, since $q_{i+1}(x)$ could be replaced by $-q_{i+1}(x)$ and the expressions corresponding to (10) and (11) would hold.

Define the interval J_k as in (5). Since $f_0(x), f_1(x), \cdots, f_n(x)$ have no common zero in J_k, $M_n(u_1, \cdots, u_n)$ has rank n at every point of J_k. Thus, by the method employed in the proof of Theorem 1, we can define a function $P_k(x)$, $k=1, 2, \cdots, m+1$, which is nonzero and of class $C^{(t-n)}$ in J_k, which satisfies the conditions

$$\begin{aligned} P_k^{(j)}(c_{k-1}+\epsilon+0) &= Q_{k-1}^{(j)}(c_{k-1}+\epsilon-0), \\ P_k^{(j)}(c_k-\epsilon-0) &= Q_k^{(j)}(c_k-\epsilon+0) \end{aligned} \tag{12}$$

for $j=0, 1, \cdots, t-n$, where

$$\begin{aligned} Q_0^{(j)}(c_0+\epsilon-0) &= Q_1^{(j)}(c_1-\epsilon+0), \\ Q_{m+1}^{(j)}(c_{m+1}-\epsilon+0) &= Q_m^{(j)}(c_m+\epsilon-0), \end{aligned}$$

and for which the differential equation $W(u_1, \cdots, u_n, y)=P_k(x)$, $k=1, 2, \cdots, m+1$, has a solution of class $C^{(t)}$ defined in J_k.

In J_1 the differential equation

$$W(u_1, \cdots, u_n, y) = P_1(x) \tag{13}$$

has a solution $p_1(x)$ with $p_1^{(j)}(c_1-\epsilon-0)=q_1^{(j)}(c_1-\epsilon+0)$, $j=0, 1, \cdots, t$, in view of the second set of conditions in (12). Then the function

$$y_1(x) \equiv \begin{cases} p_1(x), & a \leqq x \leqq c_1-\epsilon, \\ q_1(x), & c_1-\epsilon \leqq x \leqq c_1+\epsilon, \end{cases}$$

is of class $C^{(t)}$, and $M_s(u_1, \cdots, u_n, y_1)$ has rank $n+1$ at every point of the interval $a \leqq x \leqq c_1+\epsilon$ since $p_1(x)$ is a solution of (13) and the determinant Δ_{1,r_1} does not vanish in the interval $c_1-\epsilon \leqq x \leqq c_1+\epsilon$ by (10).

In J_2 the differential equation

$$W(u_1, \cdots, u_n, y) = P_2(x) \tag{14}$$

has a solution $p_2(x)$ with $p_2^{(j)}(c_1+\epsilon+0)=y_1^{(j)}(c_1+\epsilon-0)$, $j=0, 1, \cdots, t$, and also a solution $r(x)$ with $r^{(j)}(c_2-\epsilon-0)=q_2^{(j)}(c_2-\epsilon+0)$, $j=0, 1, \cdots, t$. Since $u_1(x), u_2(x), \cdots, u_n(x)$ are linearly independent solutions of the homogeneous equation corresponding to (14), for some set of constants $a_1, a_2, \cdots, a_n$,

$$p_2(x) \equiv r(x) + a_1u_1(x) + \cdots + a_nu_n(x), \quad c_1+\epsilon \leqq x \leqq c_2-\epsilon.$$

Then the function

$$y_2(x) \equiv \begin{cases} p_2(x), & c_1+\epsilon \leqq x \leqq c_2-\epsilon, \\ q_2(x) + a_1u_1(x) + \cdots + a_nu_n(x), & c_2-\epsilon \leqq x \leqq c_2+\epsilon, \end{cases}$$

is of class $C^{(t)}$ in its interval of definition with

$$y_2^{(j)}(c_1+\epsilon+0) = y_1^{(j)}(c_1+\epsilon-0), \qquad j = 0, 1, \cdots, t,$$

and $M_s(u_1, \cdots, u_n, y_2)$ has rank $n+1$ throughout the interval $c_1+\epsilon \leqq x \leqq c_2+\epsilon$, since $p_2(x)$ is a solution of (14) and the determinant Δ_{2,r_2} does not vanish in the interval $c_2-\epsilon \leqq x \leqq c_2+\epsilon$ by (10).

We continue in the above manner, obtaining $y_k(x)$, $c_{k-1}+\epsilon \leqq x \leqq c_k+\epsilon$ ($k=2, 3, \cdots, m+1$, where $c_{m+1}+\epsilon=b$) of class $C^{(t)}$ in its interval of definition with

$$y_k^{(j)}(c_{k-1}+\epsilon+0) = y_{k-1}^{(j)}(c_{k-1}+\epsilon-0), \qquad j = 0, 1, \cdots, t,$$

and with each matrix $M_s(u_1, \cdots, u_n, y_k)$ of rank $n+1$ throughout the interval $c_{k-1}+\epsilon \leqq x \leqq c_k+\epsilon$. Then the function $u_{n+1}(x)$ defined in J by

$$u_{n+1}(x) \equiv y_k(x), \quad c_{k-1}+\epsilon \leqq x \leqq c_k+\epsilon, \qquad k = 1, 2, \cdots, m+1,$$

is of class $C^{(t)}$ in J and the Wronskian matrix $M_s(u_1, \cdots, u_{n+1})$ has rank $n+1$ throughout J.

4. Application to the theory of differential equations. The extension property of Wronskian matrices leads us to the following sufficient condition that a set of n given functions be solutions of an ordinary homogeneous linear differential equation.

THEOREM 3. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ $(t \geqq n)$ in an interval J and their Wronskian matrix of order s $(s<t)$ has rank n at every point of J, then the u's are linearly independent solutions of a homogeneous linear differential equation of order $s+1$ of the type*

$$y^{(s+1)} + f_s(x)y^{(s)} + \cdots + f_1(x)y' + f_0(x)y = 0 \tag{15}$$

in which the $f_i(x)$ are functions of class $C^{(t-s-1)}$ in J.

PROOF. Let $m=s-n+1$. The case $m=0$ is well known and $m<0$ is impossible. Assume now that $m>0$. Then $n \leqq s<t$ and, by m successive applications of Theorem 2, there exist m functions, $u_{n+1}(x), u_{n+2}(x), \cdots, u_{n+m}(x)$, of class $C^{(t)}$ in J such that the Wronskian matrices

$$M_s(u_1, \cdots, u_{n+1}), M_s(u_1, \cdots, u_{n+2}), \cdots, M_s(u_1, \cdots, u_{n+m}) \tag{16}$$

have the respective ranks $n+1, n+2, \cdots, n+m$ at every point of J. Since $n+m=s+1$, the last matrix in (16) is the Wronskian matrix of order s for $s+1$ functions of class $C^{(t)}$ $(t>s)$ with $W(u_1, \cdots, u_{s+1}) \neq 0$ throughout J. Therefore $u_1(x), u_2(x), \cdots, u_{s+1}(x)$ constitute a fundamental system of solutions of a homogeneous equation of type (15) with the $f_i(x)$ of class $C^{(t-s-1)}$ in J. Hence the n given functions have the asserted property.

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ON HARMONIC AND ANALYTIC FUNCTIONS

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If we study the behavior of a harmonic function on the boundary of the unit circle along an arc $\alpha<\theta<\beta$, it is sometimes of advantage, if the function behaves in the simplest possible way outside this arc. This problem of *isolating* the singular arc can easily be solved for a harmonic function which is bounded in the unit circle. For such a function can be expressed by means of a Poisson integral

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(\theta-\vartheta)+r^2}u(1,\vartheta)d\vartheta,$$

and then

$$v(r,\theta)=\frac{1}{2\pi}\int_{\alpha}^{\beta}\frac{1-r^2}{1-2r\cos(\theta-\vartheta)+r^2}u(1,\vartheta)d\vartheta$$

is known *to behave in the same way* as $u(r,\theta)$ in the neighborhood of the arc (α,β)—in fact the difference of the two functions tends uniformly to zero inside the arc—and $v(r,\theta)$ can be extended so as to make it harmonic and equal to zero on the rest of the circumference.

It is equally easy to solve the problem for a harmonic function $u(r,\theta)$ which is $O(1/(1-r)^n)$ near the circumference and is, therefore, the $(n+2)$nd derivative of a harmonic function, bounded in $r\leqq 1$.

The purpose of this paper is to show that the problem can be solved for any function harmonic in $r<1$. The result can be generalized to any domain which can be represented conformally on the unit circle.

THEOREM. *If $u(r,\theta)$ is a function, harmonic in the unit circle, then, given the arc $(r=1,\ \alpha<\theta<\beta)$, there is a function $v(r,\theta)$ harmonic in $r<1$, such that $u(r,\theta)-v(r,\theta)$ can be extended across the arc (α,β) so as to make it harmonic and zero on the arc; and $v(r,\theta)$ is harmonic and zero along the rest of the circumference.*

PROOF. (i) If

$$u(r,\theta)=\sum_{1}^{\infty}r^n(a_n\cos n\theta+b_n\sin n\theta),$$

we can find a nonincreasing sequence $\epsilon_n\to 0$, such that

$$|a_n|,\ |b_n|\leqq((1/2)e^{\epsilon_n n}-2)/n^2. \tag{1}$$

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We have indeed

$$r^n \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} d\theta.$$

If we put $r = e^{-\epsilon}$, we find that whatever $\epsilon > 0$ we select, $|a^n|, |b^n| \leqq Ue^{\epsilon n}$ for all n. Hence

$$\limsup \frac{\log |a_n|}{n} \leqq 0, \qquad \limsup \frac{\log |b_n|}{n} \leqq 0$$

and, therefore,

$$\epsilon_n = \max_{k=n, n-1, \cdots} [0, k^{-1} \log (2|a_k| \cdot k^2 + 4), k^{-1} \log (2|b_k| \cdot k^2 + 4)]$$

will satisfy our requirements.

(ii) There exists an entire function $\phi(z)$ such that $\phi(z) = O(e^{\epsilon|z|})$ for any $\epsilon > 0$ and sufficiently large $|z|$, $\overline{\phi(z)} = \phi(\bar{z})$, $\phi(z) = \phi(-z)$, $\phi(0) = 0$ and

$$(|a_n| + |b_n|)/\phi(n) \leqq 2/n^2, \tag{2}$$

for all n so large that $2\epsilon_n \leqq 1$. We define m_n as the least integer such that $\sum_{k>m_n} n^{2k}/(2k)! \leqq 1$. Then

$$\phi(z) = \sum_1^{m_1} \frac{(\epsilon_1 z)^{2k}}{(2k)!} + \sum_{m_1+1}^{m_2} \frac{(\epsilon_2 z)^{2k}}{(2k)!} + \cdots = \sum_{k=1}^{\infty} \frac{(\delta_k z)^{2k}}{(2k)!}.$$

This function satisfies our requirements. Evidently, it is sufficient to show that (2) is satisfied.

Since

$$\sum_{k>m_n}^{\infty} \frac{(\delta_k n)^{2k}}{(2k)!} \leqq \epsilon_{n+1}^{2m_n} \sum_{k>m_n}^{\infty} \frac{n^{2k}}{(2k!)} \leqq \epsilon_{n+1}^{2m_n},$$

we have

$$\phi(n) \geqq \sum_1^{m_n} \frac{(\delta_k n)^{2k}}{(2k)!} - \epsilon_n^{2m_n} \geqq \sum_1^{\infty} \frac{(\epsilon_n n)^{2k}}{(2k)!} - 2\epsilon_n^{2m_n}$$

$$= (1/2)(e^{\epsilon_n n} + e^{-\epsilon_n n}) - 1 - 2\epsilon_n^{2m_n},$$

and, for $2\epsilon_n \leqq 1$, this yields, by (1),

$$|a_n| n^2, \ |b| n^2 \leqq \phi(n).$$

(iii) From (2) it follows that

$$U(r, \theta) = \sum_1^{\infty} r^n \frac{a_n \cos n\theta + b_n \sin n\theta}{\phi(n)}$$

is a harmonic function, continuous in the closed unit circle. Hence we can express it by means of a Poisson integral

$$U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} U(1,\vartheta)\frac{(1-r^2)d\vartheta}{1-2r\cos(\vartheta-\theta)+r^2}$$
$$= \frac{1}{2\pi}\left(\int_\alpha^\beta + \int_{(0,2\pi)-(\alpha,\beta)}\right)U(1,\vartheta)\frac{(1-r^2)d\vartheta}{1-2r\cos(\vartheta-\theta)+r^2}$$
$$= V(r,\theta) + W(r,\theta).$$

The harmonic function $V(r,\theta)$ expressed by the first integral can be extended across the arc complementary to (α,β), so that it becomes harmonic and zero on the arc. For $W(r,\theta)$ the same is true with respect to the arc (α,β) itself. We see that the function $V(r,\theta)$ bears to $U(r,\theta)$ the same relation as the function $v(r,\theta)$, required by the theorem, should bear to $u(r,\theta)$.

(iv) Wigert's theorem asserts that if $\phi(z)$ is an entire function satisfying the first condition in (ii), then

$$f(z) = \sum_0^\infty \phi(n)z^n$$

is a function having its only singular point at $z=1$. We get easily

$$f(e^w) = \sum_{k=1}^\infty \frac{(\delta_k)^{2k}}{(2k)!}\frac{d^{2k}}{dw^{2k}}\left(\frac{1}{1-e^w}\right).$$

In a proof of the above theorem, given by Bieberbach,[1] it is shown that this last series converges uniformly in a neighborhood of any point different from $w=0$. Therefore, we have

$$f(e^{-w}) = \sum_1^\infty \frac{\delta_k^{2k}}{(2k)!}\frac{d^{2k}}{dw^{2k}}\left(\frac{1}{1-e^{-w}}\right) = -\sum_1^\infty \frac{\delta_k^{2k}}{(2k)!}\frac{d^{2k}}{dw^{2k}}\left(\frac{1}{1-e^w}\right)$$
$$= -f(e^w).$$

For $w=i\theta$, we get $(1/2)(f(e^{i\theta})+f(e^{-i\theta}))=R[f(e^{i\vartheta})]=0$. Hence $K(r,\theta)=\sum_1^\infty r^n\phi(n)\cos n\theta$ is a function, harmonic in the whole plane except at $r=1$, $\theta=0$, and is zero for $r=1$, $\theta\neq 0$.

(v) Using Parseval's identity, we find without difficulty that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K(r,\theta-x)U(1,x)dx = \sum r^n(a_n\cos n\theta + b_n\sin n\theta) = u(r,\theta).$$

[1] *Lehrbuch der Funktionentheorie*, vol. 2, pp. 288–292.

Further

$$v(r, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} K(r, \theta - x)V(1, x)dx = \frac{1}{\pi}\int_{\alpha}^{\beta} K(r, \theta - x)V(1, x)dx$$

is clearly a function, harmonic in $r<1$ and it is easy to deduce from the properties of $K(r, \theta)$ and $V(1, \theta)$ that $v(r, \theta)$ satisfies all the conditions of the theorem.

COROLLARY. *If $f(z)$ is analytic in $|z|<1$, then, given the arc ($|z|=1$, $\alpha<\arg z<\beta$), there is a function $g(z)$ analytic in $|z|<1$ and on the arc of $|z|=1$ complementary to (α, β), such that $f(z)-g(z)$ can be extended analytically across (α, β).*

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ON THE COMPLEX ZEROS OF THE BESSEL FUNCTIONS

E. HILLE AND G. SZEGÖ

1. **Introduction.** Various proofs have been given for the following classical theorem of A. Hurwitz:

THEOREM 1. *The entire function*

$$z^{\beta/2}J_{-\beta}(2z^{1/2}) = \sum_{m=0}^{\infty} \frac{(-z)^m}{m!} \frac{1}{\Gamma(m+1-\beta)} \tag{1.1}$$

has precisely $[\beta]$ nonpositive zeros. Here $J_{-\beta}$ is the Bessel function of order $-\beta$ and $\beta \geqq 0$.

In case β is an integer these nonpositive zeros are all at the origin; in case β is not an integer and $[\beta]$ is odd there is precisely one negative zero and we have $\frac{1}{2}([\beta]-1)$ pairs of conjugate complex zeros; in case β is not an integer and $[\beta]$ is even there are $\frac{1}{2}[\beta]$ pairs of conjugate complex zeros.

Most of the proofs for this theorem (see the papers [2, 4, 6, 7, 9] of the Bibliography at the end of the text) make use of polynomial approximations of the Bessel function. The present proof follows the same line by obtaining the Bessel function as the limit of *Laguerre polynomials*. The study of the complex zeros of these polynomials is

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particularly simple and for this reason the present arrangement may have certain advantages.[1] Our proof yields also certain bounds for the nonpositive zeros of the function (1.1) in terms of β.

In §2 we give a compilation of the definitions and results on Laguerre polynomials which are essential for our proof. We follow the notation of G. Szegö, *Orthogonal polynomials* (Amer. Math. Soc. Colloquium Publications, vol. 23, New York, 1939; quoted as Sz.).

§3 contains certain bounds for the nonpositive zeros of Laguerre polynomials from which Theorem 1 follows immediately.

§4 gives bounds for the nonpositive zeros of the function (1.1).

In §3 we use as an important instrument an elementary formula involving a solution of a second order linear differential equation which was employed for the study of the zeros of various functions by A. Hurwitz and E. Hille [5, 3].

2. **Laguerre polynomials.** We define the Laguerre polynomials by [Sz. (5.1.6)]

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} C_{n+\alpha,n-m} \frac{(-x)^m}{m!}. \tag{2.1}$$

This is equivalent to the formula [Sz. (5.1.5)]

$$e^{-x}x^{\alpha}L_n^{(\alpha)}(x) = (n!)^{-1}(d/dx)^n(e^{-x}x^{n+\alpha}). \tag{2.2}$$

It is easy to verify the differential equation [Sz. (5.1.2)]

$$\begin{cases} w'' + G(x)w = 0, & w = w(x) = e^{-x/2}x^{(\alpha+1)/2}L_n^{(\alpha)}(x), \\ G(x) = Ax^{-1} - Bx^{-2} - 1/4, & A = n + (\alpha+1)/2, \\ & B = (\alpha^2 - 1)/4. \end{cases} \tag{2.3}$$

Let $x = u + iv \neq 0$; we note that

$$\Im\{G(x)\} = v\{-A(u^2+v^2)^{-1} + 2Bu(u^2+v^2)^{-2}\}. \tag{2.4}$$

Finally we conclude from (2.1) by comparing the corresponding powers of x [Sz. (5.1.14)]

$$(d/dx)L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x). \tag{2.5}$$

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[1] The proof given by Pólya [7] deduces Theorem 1 from certain theorems of a general character. It may be of interest to observe that his theorem on the so-called Jensen polynomials [7, p. 165] furnishes the result of Hurwitz; indeed, the Jensen polynomials associated with the function (1.1) are (except for a factor independent of x) the Laguerre polynomials $L_n^{(-\beta)}(x)$.

LEMMA 1. *Let $n \geqq 1$, α an arbitrary real number, $\alpha \neq -1, -2, \cdots, -n$. Then the zeros of $L_n^{(\alpha)}(x)$ are not equal to 0 and distinct. Furthermore the number of the positive zeros is n if $\alpha > -1$; it is $n+[\alpha]+1$ if $-n < \alpha < -1$; it is 0 if $\alpha < -n$. The number of the negative zeros is* 0 *or* 1. [Sz., Theorem 6.73.]

Let $\alpha = -\beta$ be negative, β not an integer. If $n > \beta$ the polynomial $L_n^{(-\beta)}(x)$ has $[\beta]$ nonpositive zeros which are all complex, except a single negative one in case $[\beta]$ is odd.

This theorem is due to E. Stridsberg [8] and W. Hahn [1]. It follows from older results on Jacobi polynomials by a limiting process [Sz. loc. cit.]. A continuity proof based on the variation of β is also possible.

For the sake of completeness we give here the following very simple proof of Lemma 1.

From (2.1) and (2.3) we conclude that all the zeros are not equal to 0 and distinct. In case $\alpha > -1$ Rolle's theorem applied to formula (2.2) furnishes exactly n positive zeros. Further let $-k-1 < \alpha < -k$ $(1 \leqq k \leqq n)$. The same theorem furnishes then at least $n-k$ positive zeros, and by Descartes' rule of signs we obtain exactly this number of positive zeros. Inspection of (2.1) shows that for $k \geqq n$ no positive zeros exist. Now let x be negative and $\alpha < -1$. Then[2] $G(x) < 0$ so that w and w'' are of the same sign. This excludes the possibility of two negative zeros. Hence the number of the negative zeros is 0 or 1.

LEMMA 2. *Let α be an arbitrary real number. Then*

$$\lim_{n\to\infty} n^{-\alpha} L_n^{(\alpha)}(z/n) = z^{-\alpha/2} J_\alpha(2z^{1/2}) \tag{2.6}$$

uniformly if z is bounded. [Sz. Theorem 8.1.3.]

This follows immediately from (2.1).

In view of the limit formula (2.6)Hurwitz's theorem results, provided we can show that the nonpositive zeros of $L_n^{(\alpha)}(x)$ are in absolute value not greater than Cn^{-1}, where $C = C(\alpha)$ is independent of n. (Indeed, the zeros of the functions on the left-hand side of (2.6) tend to those of the function on the right-hand side, as $n \to \infty$.) The only possibility which has to be excluded is the *confluence* of two conjugate complex zeros of $L_n^{(\alpha)}(x)$ into a real zero of $z^{-\alpha/2} J_\alpha(2z^{1/2})$, which would be necessarily multiple and not equal to 0. But this is out of the question since this function as a solution of a second order linear differ-

[2] $G(x) \leqq (\alpha+1)/2x - (\alpha^2-1)/4x^2 - 1/4 \leqq (1/2)/(\alpha-1) < 0$.

ential equation with the only finite singularity $z=0$ can not have a multiple zero different from 0.

3. Bounds for the nonpositive zeros of $L_n^{(\alpha)}(x)$. In what follows we use the notation of §2.

LEMMA 3. *Let $\alpha=-\beta$ be negative, β not an integer and $[\beta]$ odd. Then for the only negative zero x_0 of $L_n^{(-\beta)}(x)$ we have*

$$x_0 > -\beta^2/4n \tag{3.1}$$

provided n is sufficiently large.

According to (2.5) $L_n^{(-\beta)}(x)$ is monotonic for negative x; furthermore $L_n^{(-\beta)}(0)<0$. Thus $x_0>-cn^{-1}$, $c>0$, provided $L_n^{(-\beta)}(-cn^{-1})>0$. Since, $n>[\beta]$,

$$\begin{aligned} L_n^{(-\beta)}(-cn^{-1}) &= \sum_{m=0}^{[\beta]} C_{n-\beta,n-m}\frac{(cn^{-1})^m}{m!} + \sum_{m=[\beta]+1}^{n} C_{n-\beta,n-m}\frac{(cn^{-1})^m}{m!} \\ &> \sum_{m=0}^{[\beta]} C_{n-\beta,n-m}\frac{(cn^{-1})^m}{m!} \end{aligned} \tag{3.2}$$

and

$$\lim_{n\to\infty} n^{\beta}\sum_{m=0}^{[\beta]} C_{n-\beta,n-m}\frac{(cn^{-1})^m}{m!} = \sum_{m=0}^{[\beta]}\frac{c^m}{m!}\,\frac{1}{\Gamma(m+1-\beta)}, \tag{3.3}$$

we have $L_n^{(-\beta)}(-cn^{-1})>0$ for sufficiently large n provided the expression on the right-hand side of (3.3) is positive. This is the case if c is chosen sufficiently large.

It is easy to specify the constant c. The above expression is certainly positive provided

$$c > m(\beta - m), \qquad m = 1, 2, 3, \cdots, [\beta]. \tag{3.4}$$

We can take, for instance,

$$c = \beta^2/4. \tag{3.5}$$

The formula of A. Hurwitz and E. Hille mentioned in the introduction applied to (2.3) will read as follows:

LEMMA 4. *Let a and b be arbitrary complex numbers, $a\neq b$, $a\neq 0$, $b\neq 0$. Then*

$$[\bar{w}w']_a^b - \int |w'|^2\overline{dx} + \int G(x)|w|^2dx = 0; \tag{3.6}$$

the integration is extended over an arbitrary rectifiable curve in the com-

plex x-plane joining a and b and avoiding the origin.

This furnishes immediately a bound for the imaginary parts of the zeros of $L_n^{(\alpha)}(x)$. More exactly we prove:

LEMMA 5. *Let $\alpha=-\beta$ be negative, β not an integer, $n>\beta>1$. For any complex zero $p+iq$ of $L_n^{(-\beta)}(x)$ we have*

$$|q| \leqq B/A. \tag{3.7}$$

Let $q\neq 0$; we apply (3.6), integrating from $p+iq$ to ∞ along a straight line parallel to the real axis with $\Re x\to+\infty$. Then, noting that $\overline{w(x)}w'(x)\to 0$ when $x=u+iq$, $u\to+\infty$,

$$-\int_{p+iq}^{\infty} |w'|^2dx+\int_{p+iq}^{\infty} G(x)\,|w|^2dx=0$$

follows, so that $\Im\{G(x)\}$ must vanish somewhere along this path. Thus we must have $-A(u^2+v^2)+2Bu=0$ for a suitable $u, u\geqq p, v=q$. This is the equation of a circle with center at $(B/A, 0)$ and radius B/A. From this (3.7) follows.

LEMMA 6. *Let $\alpha=-\beta$ be negative, β not an integer, $n>\beta>1$. Then for the complex zeros $p+iq$ of $L_n^{(-\beta)}(x)$ we have*

$$|p| \leqq \begin{cases} 2B/A & \text{if } p>0, \\ |x_0| & \text{if } p<0, \end{cases} \tag{3.8}$$

where x_0 is the only negative zero of $L_n^{(-\beta)}(x)$ or $L_{n+1}^{(-\beta-1)}(x)$ depending upon whether $[\beta]$ is odd or even.

The previous argument furnishes the first part of the assertion. As to the second part, we assume that $[\beta]$ is odd; the case in which $[\beta]$ is even can be settled by means of Gauss' theorem on account of the relation

$$(d/dx)L_{n+1}^{(-\beta-1)}(x) = -L_n^{(-\beta)}(x).$$

Let us assume that $p<x_0$, $q>0$. We apply (3.6) with $a=p$, $b=p+iq$ choosing for the path of integration a vertical segment. Then

$$-\overline{w(p)}w'(p)-\int_p^{p+iq} |w'|^2\overline{dx}+\int_p^{p+iq} G(x)\,|w|^2dx=0 \tag{3.9}$$

follows. We write

$$\mathfrak{w}(x) = e^{-x/2}(-x)^{(\alpha+1)/2}L_n^{(\alpha)}(x) \tag{3.10}$$

and take the real parts of the terms in (3.9):

$$(3.11)\quad -\{\mathfrak{w}(p)\}^2\frac{\mathfrak{w}'(p)}{\mathfrak{w}(p)}-\int_p^{p+iq}\{\Im G(x)\}\,|\mathfrak{w}(x)|^2|\,dx|=0.$$

Now $\mathfrak{w}(x)$ satisfies the differential equation (2.3) and $G(x)<0$ for $x<0$. Hence $\mathfrak{w}''(x)$ and $\mathfrak{w}(x)$ are of the same sign, that is $\mathfrak{w}''(x)>0$ for $x<x_0$. Thus $\mathfrak{w}'(x)$ is increasing for $x<x_0$ and since $\mathfrak{w}'(x_0)=e^{-x_0/2}(-x_0)^{(\alpha+1)/2}L_n^{(\alpha)\prime}(x_0)<0$ we have $\mathfrak{w}'(x)<0$ for $x<x_0$. Consequently the first term of (3.11) is positive. But according to (2.4) $\Im\{G(x)\}<0$ so that (3.11) involves a contradiction.

4. Conclusion. From the Lemmas 3, 5 and 6 we deduce Theorem 1 in a familiar way (cf. the end of §2). By using (3.1), (3.7) and (3.8) we find more precisely:

THEOREM 2. *Let $\alpha=-\beta$ be negative, β not an integer, $\beta>1$. The non-positive zeros of the function $z^{\beta/2}J_{-\beta}(2z^{1/2})$ are situated in the rectangle*

$$(4.1)\quad \left.\begin{matrix}-\beta^2/4\\ -(\beta+1)^2/4\end{matrix}\right\}\leqq \Re z\leqq(\beta^2-1)/2;\quad |\Im z|\leqq(\beta^2-1)/4.$$

In the first inequality the upper or lower expression holds depending on whether $[\beta]$ is odd or even.

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5. ———, *Über die Nullstellen der hypergeometrischen Funktion*, Math. Ann. vol. 64 (1907) pp. 517–560. See *Mathematische Werke*, vol. 1, 1932, pp. 660–705, in particular pp. 700–704.

6. N. Obreschkoff, *Über die Nullstellen der Besselschen Funktionen*, Jber. Deutschen Math. Verein. vol. 38 (1929) pp. 156–161.

7. G. Pólya, *Über einen Satz von Laguerre*, Jber. Deutschen Math. Verein. vol. 38 (1929) pp. 161–168.

8. E. Stridsberg, *Några aritmetiska undersökningar rörande fakulteter och vissa allmännare Koefficientsviter*, Not 2, Arkiv för Matematik, Astronomi och Fysik, vol. 13 no. 25 (1918) 70 pp., in particular pp. 34–35.

9. G. N. Watson, *The zeros of Lommel's polynomials*, Proc. London Math. Soc. (2) vol. 19 (1921) pp. 266–272.

YALE UNIVERSITY AND
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ON LINEAR SPACES WHICH MAY BE RENDERED COMPLETE NORMED METRIC SPACES

CASPER GOFFMAN

In this paper, we obtain a characterization of linear spaces which may be normed so as to become complete, linear, normed metric spaces. In this connection, K. Kunugui[1] and M. Fréchet[2] have shown that every metric space S is isometric with a subset of a complete, linear, normed metric space. It follows from our result that if the cardinal number of S is the limit of a denumerable sequence of cardinals, then there is no complete, linear, normed metric space isometric with S. Results on topological spaces which may be rendered linear, normed metric spaces and complete, linear, normed metric spaces have been given by A. Kolmogoroff[3] and B. Z. Vulich.[4]

It will be assumed that the reader is familiar with certain elementary portions of the theory of linear and metric spaces, and with transfinite cardinal and ordinal numbers.[5] Using the generalized continuum hypothesis and normal order theorem, we prove the following:

THEOREM. *A necessary and sufficient condition that a linear space may be made a complete, linear, normed metric space by a suitable definition of norm is that the cardinal number of its Hamel basis should not be the limit of any denumerable sequence of cardinals which precede it.*

A Hamel basis of a linear space S is a subset T of S such that every element of S is a linear combination, with real coefficients, of a finite number of elements of T, and there is no proper subset of T with this property. The following properties of a Hamel basis will be used in demonstrating the theorem, and are given without proof:

(a) A linear space S has a Hamel basis.[6]

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[1] K. Kunugui, *Applications des espaces à une infinité de dimensions à la théorie des ensembles*, Proc. Imp. Acad. Tokyo vol. 11 no. 9 pp. 351–353.

[2] M. Fréchet, *Sur les espaces distanciés*. Memorial volume dedicated to D. A. Grave. Moscow, 1940, pp. 265–267.

[3] A. Kolmogoroff, *Zur Normierbarkeit eines allgemeinen topologischen Raumes*, Studia Mathematica, vol. 5 (1934) pp. 29–33.

[4] B. Z. Vulich, *Linear spaces with given convergence*, Leningrad State University Annals, Mathematics Series vol. 10 (1940) pp. 40–63.

[5] For the elements of transfinite number theory, see F. Hausdorff, *Mengenlehre*, Berlin, 1927; for the theory of linear spaces, see S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

[6] Proofs of (a) and (b) have been given by H. Löwig, *Über die Dimension linearer Räume*, Studia Mathematica, vol. 5 (1934) pp. 18–24.

(b) S may have many distinct bases, but these must have the same cardinal number.

(c) If the basis T of S is nondenumerable, then s and t, the respective cardinals of S and T, are identical.

(d) The representation of every element x of S as a linear combination, with nonzero coefficients, of a finite number of elements of a given basis T of S is unique.

1. **Sufficiency.** We note that a linear, normed metric space with a finite Hamel basis, of cardinal n, is homeomorphic with euclidean n-space; hence, is complete. To prove the sufficiency part of the theorem we need, therefore, consider only the infinite case; that is, we must show that a linear space whose basis is of infinite cardinal not the limit of any denumerable sequence of cardinals which precede it may be rendered complete by a suitable definition of norm. Such cardinals are either non-limiting, infinite cardinals, or limiting cardinals which are not the limit of any denumerable sequence of cardinals. First, let α be a non-limiting, infinite cardinal, and β its immediate predecessor in the normal order of cardinals. By the generalized continuum hypothesis, $2^\beta=\alpha$. Let S be the linear space of all bounded, real functions defined on a given set of cardinal β, the sum of two functions and the product of a function by a real number being defined in the customary way. S is of cardinal $c^\beta=2^\beta=\alpha$. With $\nu(f)=$l.u.b. $|f(x)|$, the least upper bound of $|f(x)|$, as norm, S is evidently a complete, linear, normed metric space. For $\alpha>c$, as previously noted, every basis of S has the same cardinal α as S. For $\alpha=c$, S certainly has no finite basis and, since complete, has no denumerable basis, as will be shown in the proof of the necessity part of the theorem. Hence, by the continuum hypothesis, if S is of cardinal c, its basis is also of cardinal c. We have thus defined, for every non-limiting, infinite cardinal α, a complete, linear, normed metric space with basis of cardinal α. Any linear space S' with basis of cardinal α is isomorphic with S. For, if T and T' are respective bases of S and S', a biunique correspondence exists between them. It then follows from the uniqueness of the representation of the elements of a space as a linear combination of basal elements, that S and S' are isomorphic linear spaces. Hence, by letting the norm of every element of S' be identical with that of its mate in S, the linear space S' is made complete. Now, let α be a limiting, infinite cardinal which is not the limit of any denumerable sequence of cardinals which precede it. For every cardinal $\beta<\alpha$, let T_β be a set of cardinal β such that, for every $\gamma<\beta$, T_γ is a subset of T_β. Such a class of sets may be obtained, for ex-

ample, by normally ordering a set of cardinal α, and letting T_β be the proper initial segment of the order which introduces the cardinal β. Let $T=\sum_{\beta<\alpha}T_\beta$ and let S_β be the set of bounded, real functions defined over T, arbitrarily on T_β and identically zero on $T-T_\beta$. The cardinal number of S_β is the immediate successor of β, by the generalized continuum hypothesis, and the cardinal number of $S=\sum_{\beta<\alpha}S_\beta$ is α. As in the preceding case, S may be rendered a linear, normed metric space by defining the sum of two functions of S and the product of a function of S by a real number in the manner customary for functions, and by letting $\nu(f)=$l.u.b. $|f(x)|$, for every $f(x)$ in S. The space S is a complete metric space. For, let $\{f_n(x)\}$ be any convergent sequence of elements of S. Then $\{f_n(x)\}$ is a uniformly convergent sequence of functions. Let β_n be the smallest cardinal such that $f_n(x)$ is identically zero on the set $T-T_{\beta_n}$. Then $\beta_n<\alpha$ and, since α is not the limit of any denumerable sequence of cardinals which precede it, there exists a cardinal $\beta<\alpha$ such that $\beta_n<\beta$, for all n. The functions of the sequence $\{f_n(x)\}$ are thus identically zero on $T-T_\beta$ and, since the sequence is convergent, its limit, $f(x)$, is also identically zero on $T-T_\beta$. Moreover, $f(x)$ is bounded, for it is the limit of a uniformly convergent sequence of bounded functions. Hence, $f(x)$ is in S_β and thus in S. The linear, normed metric space S is therefore complete. It then follows from the isomorphism of linear spaces whose bases are of the same cardinal, that every linear space with basis of cardinal α may be normed so as to be complete.

2. **Necessity.** We show that a linear, normed metric space is not a complete metric space if the cardinal number of its basis is the limit of a denumerable sequence of cardinals which precede it. We prove the following:

LEMMA. *If T is a nondense, linear subspace of a linear, normed metric space S, then T is nowhere dense in S.*

PROOF. Let σ be a sphere of elements of S. Then $c\sigma$, the set obtained by multiplying the elements of σ by a real number $c\neq 0$, and $x+\sigma$, the set obtained by adding to the elements of σ a given element x of S, are also spheres in S. Since T is nondense in S, there is a sphere σ in S, containing no point of T. Let $\{c_n\}$ be a sequence of reals converging to zero. Then, for every x in T, the sequence $\{x+c_n\sigma\}$ of spheres converges to x. But these spheres contain no points of T. Hence, T is nowhere dense in S.

Suppose S is a linear, normed metric space with a denumerable basis, whose elements are $\xi_1, \xi_2, \cdots, \xi_n, \cdots$. Let S_n be the linear

subspace of S with basis $\xi_1, \xi_2, \cdots, \xi_n$. Then $S=\sum_1^\infty S_n$. Since S_n has a finite basis, it is a complete metric space; hence, closed in S. But a closed subset of a metric space is nondense in the space. Therefore, by the lemma, S_n is nowhere dense in S, and S, as the limit of $\aleph_0$ nowhere dense sets, is of the first category, and therefore not complete.

Finally, we construct, for every nondenumerable cardinal α which is the limit of a denumerable sequence of cardinals, a linear space of cardinal α, and hence, with basis of cardinal α, which cannot be normed so as to be rendered complete. Since, as noted above, two linear spaces are isomorphic if their bases are of the same cardinal, it follows that every linear space of cardinal α has the desired property. Now, let $\{\alpha_n\}$ be the sequence of consecutive, increasing cardinals which converges to α, and let T_n be a set of cardinal α_n such that, for every n, T_n is a subset of T_{n+1}. Let $T=\sum_1^\infty T_n$. Let S_n be the set of bounded, real functions defined over T, arbitrarily on T_n and zero elsewhere, and let $S=\sum_1^\infty S_n$. The cardinal number of S_n is $2^{\alpha_n}=\alpha_{n+1}$, by the generalized continuum hypothesis, and the cardinal number of S is $\alpha=\sum_1^\infty \alpha_{n+1}$. S is rendered a linear space, if the sum of two functions and the product of a function by a real number are defined in the usual way. Suppose S is further rendered normed and metric by any given choice of norm for the elements of S. S_n is obviously a linear, normed metric proper subspace of S. Moreover, S_n is a nondense subset of S. For there are only $\alpha_{n+1}^{\aleph_0}=\alpha_{n+1}$ sequences composed of elements of S_n and, since S is of cardinal $\alpha>\alpha_{n+1}$, there exists an element of S not a limit point of S_n. By the lemma, S_n is nowhere dense in S and $S=\sum_1^\infty S_n$ is of the first category. Hence S is not complete.

As a consequence of the necessity part of this theorem, it immediately follows that a metric space whose cardinal number is the limit of no denumerable sequence of cardinals is homeomorphic, hence isometric, with no complete, linear, normed metric space.

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THE STRONGER FORM OF CAUCHY'S INTEGRAL THEOREM

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1. Introduction. The so-called weaker and stronger forms[1] of Cauchy's integral theorem are the following.

THEOREM 1. CAUCHY'S INTEGRAL THEOREM (WEAKER FORM). *If $f(z)$ is holomorphic on the finite simply-connected open domain D, and if C is a closed rectifiable curve in D, then*

$$\int_C f(z)dz = 0.$$

THEOREM 2. CAUCHY'S INTEGRAL THEOREM (STRONGER FORM). *If $f(z)$ is holomorphic on the interior D of a simply closed rectifiable curve C, and continuous on $D+C$, then*

$$\int_C f(z)dz = 0. \tag{1}$$

Each of these theorems follows readily when it has been established that there is a sequence of closed curves $\{C_n\}$ in D, of uniformly bounded length, converging in the sense of Fréchet to C, such that

$$\int_{C_n} f(z)dz = 0.$$

In the case of Theorem 2 it appears to be more difficult to establish the existence of the sequence $\{C_n\}$, since the convergence must be from only one side; but the difficulty may be overcome by more or less tedious topological considerations, a program which has been undertaken by a number of authors.[2] Recent excellent proofs by Reid and Hestenes, loc. cit., appear to be about as simple as a valid elementary proof of this result could be.

The above type of proof applies equally well to yield the corresponding stronger form of Green's lemma, as some authors have

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[1] This terminology is taken from M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge, England, 1939, pp. 154, 156.

[2] For bibliographical discussions see S. Pollard, *On the conditions for Cauchy's theorem*, Proc. London Math. Soc. vol. 21 (1923) pp. 456–482; E. Kampe, *Zu dem Integralsatz von Cauchy*, Math. Zeit. vol. 35 (1932) pp. 539–543; W. T. Reid, *Green's lemma and related results*, Amer. J. Math. vol. 63 (1941) pp. 563–574; M. R. Hestenes, *An analogue of Green's theorem in the calculus of variations*, Duke Math. J. vol. 8 (1941) pp. 300–311.

pointed out, and also to yield other integral results such as the analogous stronger form of Cauchy's integral formula, and so on.

Walsh and Heilbronn[3] have given brief and simple, but not elementary, proofs of Theorem 2, using results of Walsh and Fejér on polynomial boundary approximations to holomorphic functions. Their methods of proof do not yield the stronger form of Green's lemma.

As an application of Theorem 2, consider Painlevé's theorem in the following form.

PAINLEVÉ'S THEOREM. *If open plane domains D_1 and D_2 abut along the rectifiable arc C, if $f_j(z)$ is holomorphic on D_j and continuous on D_j+C, $j=1, 2$, and if $f_1(z)=f_2(z)$ on C, then $f_1(z)$ and $f_2(z)$ are analytic continuations of one another across C.*

The proofs of this theorem use the finite accessibility of points of C from D_1 and from D_2, together with the stronger form of Cauchy's integral theorem and sometimes of Cauchy's integral formula; the weaker forms do not suffice.

The following proof of Theorem 2, though not elementary, uses only tools which are usually considered standard. It is brief and simple, and the method has the advantage of being equally applicable to establishing the stronger form of Green's lemma, Cauchy's integral formula, and so on.

2. **Lemmas.** We shall use the following two lemmas.

LEMMA 1. BIEBERBACH-CSILLAG THEOREM.[4] *If $z=g(w)$ is holomorphic on $|w|<1$ and continuous on $|w|\leqq 1$, and if $l(r)$ denotes the length of the image of $|w|=r\leqq 1$ on the z-plane, then $l(r)$ is a monotonically non-decreasing function of r for $0\leqq r\leqq 1$.*

PROOF. Fix $r<1$ and let $r<R\leqq 1$. Since the length of a curve is the limit of the lengths of a proper sequence of inscribed polygons, for any positive ϵ there is a set of values $\theta_1, \theta_2, \cdots, \theta_n, \theta_{n+1}=\theta_1$, with $\theta_1<\theta_2<\cdots<\theta_n<\theta_1+2\pi$, such that

$$p(r) \equiv \sum_{j=1}^{n} \left| g(re^{i\theta_{j+1}}) - g(re^{i\theta_j}) \right| > l(r) - \epsilon. \tag{2}$$

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[3] J. L. Walsh, *The Cauchy-Goursat theorem for rectifiable Jordan curves*, Proc. Nat. Acad. Sci. U.S.A. vol. 19 (1933) pp. 540–541; H. Heilbronn, *Zu dem Integralsatz von Cauchy*, Math. Zeit. vol. 37 (1933) pp. 37–38.

[4] L. Bieberbach, *Über die konforme Kreisabbildung nahezu kreisförmiger Bereiche*, Preuss. Akad. Wiss. Sitzungsber, 1924, pp. 181–188.

Consider the function $p(w) \equiv \sum_{j=1}^{n} \left| g(we^{i\theta_{j+1}}) - g(we^{i\theta_j}) \right|$; now $p(w)$ is continuous for $|w| \leqq 1$; and since the absolute value of a holomorphic function is subharmonic and the sum of a finite number of subharmonic functions is subharmonic, it follows that $p(w)$ is subharmonic for $|w| < 1$. Hence $p(w)$ assumes its maximum value on $|w| \leqq R$ at some point $Re^{i\theta_0}$ on $|w| = R$; in particular,

$$p(r) \leqq p(Re^{i\theta_0}). \tag{3}$$

Clearly $p(Re^{i\theta_0})$ is the length of a polygon inscribed in the image of $|w| = R$, so that

$$p(Re^{i\theta_0}) \leqq l(R). \tag{4}$$

Now (2), (3) and (4) yield $l(r) - \epsilon < l(R)$, whence $l(r) \leqq l(R)$.

LEMMA 2. COROLLARY OF HELLY-BRAY THEOREM.[5] *In the unit circle $r \leqq 1$, let $g(re^{i\theta})$ and $h(re^{i\theta})$ be continuous complex functions and let $g(re^{i\theta})$, as function of θ, $0 \leqq \theta \leqq 2\pi$, be of limited variation, uniform with respect to r. Then*

$$\int_0^{2\pi} h(e^{i\theta}) dg(e^{i\theta}) = \lim_{r \to 1} \int_0^{2\pi} h(re^{i\theta}) dg(re^{i\theta}).$$

PROOF. Let T be the least upper bound of the total variations T_r. Since $h(re^{i\theta})$ is uniformly continuous in the unit circle, for any positive ϵ there is a positive integer n such that for all $r \leqq 1$,

$$\left| h(re^{i\theta}) - h(re^{i\theta'}) \right| \leqq \epsilon/(3T)$$

provided $|\theta - \theta'| \leqq 2\pi/n$. Then for all $r \leqq 1$, we have[6]

$$\left| \int_0^{2\pi} h(re^{i\theta}) dg(re^{i\theta}) - \sum_{j=1}^{n} h(re^{2j\pi i/n}) \left[g(re^{2j\pi i/n}) - g(re^{2(j-1)\pi i/n}) \right] \right| \leqq \epsilon/3. \tag{5}$$

Also, by the continuity of $g(re^{i\theta})$ and $h(re^{i\theta})$ as functions of r, there is an $r_0 < 1$ such that for $r_0 < r < 1$,

$$\left| \sum_{j=1}^{n} h(re^{2j\pi i/n}) \left[g(re^{2j\pi i/n}) - g(re^{2(j-1)\pi i/n}) \right] - \sum_{j=1}^{n} h(e^{2j\pi i/n}) \left[g(e^{2j\pi i/n}) - g(e^{2(j-1)\pi i/n}) \right] \right| < \epsilon/3. \tag{6}$$

[5] See for instance G. C. Evans, *The logarithmic potential*, Amer. Math. Soc. Colloquium Publications, vol. 6, 1927, pp. 14–16.

[6] Cf. K. Knopp, *Funktionentheorie*, vol. 1, Berlin, 1930, p. 44, Hilfssatz 2.

Now (5), taken once for $r_0<r<1$ and again for $r=1$, and (6) yield

$$\left|\int_0^{2\pi} h(e^{i\theta})dg(e^{i\theta}) - \int_0^{2\pi} h(re^{i\theta})dg(re^{i\theta})\right| < \epsilon,$$

and the lemma is established.

3. **Proof of Theorem** 2. By the Riemann mapping theorem, there is a holomorphic function $z=g(w)$ which maps the interior of the unit circle $|w|<1$ on D; further, by the Osgood-Carathéodory theorem, $g(w)$ remains continuous on $|w|\leqq 1$ and $z=g(w)$ maps $|w|=1$ topologically on C. Let $f(z)=f[g(w)]=h(w)$. Now $g(w)$ and $h(w)$ are continuous on $|w|\leqq 1$ and, by Lemma 1, $g(re^{i\theta})$, as function of θ, $0\leqq\theta\leqq 2\pi$, is of limited variation, uniform with respect to r. Consequently, by Lemma 2,

$$\begin{aligned}\int_C f(z)dz &= \int_0^{2\pi} f[g(e^{i\theta})]dg(e^{i\theta}) = \int_0^{2\pi} h(e^{i\theta})dg(e^{i\theta}) \\ &= \lim_{r\to 1}\int_0^{2\pi} h(re^{i\theta})dg(re^{i\theta}) = \lim_{r\to 1}\int_{C_r} f(z)dz,\end{aligned} \tag{7}$$

where C_r is the image of $|w|=r$ on the z-plane. But by Theorem 1,

$$\int_{C_r} f(z)dz = 0. \tag{8}$$

Now (1) follows from (7) and (8).

THE UNIVERSITY OF TEXAS

ON AUTOMORPHISMS OF COMPACT GROUPS

PAUL R. HALMOS

1. Introduction and definitions. Let G be a compact abelian group and α a continuous automorphism of G. We write G multiplicatively and use, accordingly, the exponent notation for automorphisms. Thus the image under α of the element $x \in G$ will be denoted by x^α; similarly we shall write for (complex valued) functions $f(x)$, $f^\alpha(x) = f(x^\alpha)$.[1]

If m is Haar measure[2] in G (normalized so that $m(G) = 1$) we consider the set function $m'(E) = m(E^\alpha)$. (E^α is the set of all x^α, $x \in E$.) Since m' is a measure on G possessing all defining properties of m it follows from the uniqueness of Haar measure[3] that $m'(E) = m(E)$ for every measurable set E. In other words α is a measure preserving transformation of G; the purpose of this note is to investigate a few simple properties of α from the point of view of measure theory.

We shall make use of the Pontrjagin duality theory,[4] and, in particular, we shall need the fact that the group of automorphisms of G is essentially the same as that of the character group G^*. More precisely: if to any $\phi = \phi(x) \in G^*$ we make correspond $\phi^\alpha = \phi^\alpha(x) = \phi(x^\alpha)$, then $\phi^\alpha \in G^*$, and the correspondence $\phi \to \phi^\alpha$ is an automorphism of G^*. The duality theory also enables us to reverse this argument: every automorphism of G^* is induced in this way by a continuous automorphism of G.

We recall some standard definitions from ergodic theory. A measure preserving transformation α (not necessarily an automorphism) is *ergodic* if the only (complex valued, measurable) solutions f of the equation $f^\alpha = f$ are constant almost everywhere. The transformation α is *mixing* if the only (complex valued, measurable) solutions f of the equation $f^\alpha = \lambda f$, for any constant λ, are constant almost everywhere.[5] (It is true, though irrelevant, that for $\lambda \neq 1$ even a constant fails to be a solution unless it is zero.) It is well known that the mapping

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[1] This notation dovetails, as usual, with ordinary exponentiation in G; thus $x^{3\alpha^2} = (x^3)^{\alpha^2} = (x^{\alpha^2})^3$, and so on.

[2] For a general discussion of measure theory in topological groups see A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris, 1938.

[3] Weil, op. cit., pp. 36–38.

[4] Weil, op. cit., chap. 6.

[5] See E. Hopf, *Ergodentheorie*, Berlin, 1937, chap. 3, for a discussion of the fact that these definitions are equivalent to the ones more commonly given.

$f \to f^\alpha$ induced by a measure preserving transformation α on the space of functions over G is a unitary transformation of the Hilbert space $L_2(G)$.[6] Two measure preserving transformations α and β are of the *same spectral type* if there is a unitary transformation ω of $L_2(G)$ (ω need not be induced by a transformation of G) for which $f^{\omega\alpha\omega^{-1}} = f^\beta$ for all $f \in L_2(G)$. Given any point $x \in G$ (or function f on G) the set of all x^{α^n} (or f^{α^n}), $n = 0, \pm 1, \pm 2, \cdots$ is the *orbit* of x (or f). If α is an automorphism the orbit of the identity consists of the identity only; if this is the only finite orbit we shall say, for the sake of brevity, that α has no finite orbits.

In terms of the definitions of the preceding paragraph we can state our results quite concisely. In Theorem 1 we obtain a simple characterization of ergodicity and mixing in terms of the orbits of α in G^*. Theorem 2 is a statement concerning abstract groups and, prima facie, has nothing to do with measure theory; together with Theorem 1 and the duality theory it yields, however, a complete description of the spectral type of ergodic automorphisms. In the concluding section we state some unsolved problems and emphasize the importance of group automorphisms as a source of many new and simple examples of transformations with properties that were once considered difficult to obtain.[7]

2. **Ergodic and mixing automorphisms.** We prove the following theorem:

THEOREM 1. *A continuous automorphism α of a compact abelian group G is ergodic (or mixing) if and only if the induced automorphism on the character group G^* has no finite orbits.*

We call attention to the somewhat surprising fact that for continuous automorphisms ergodicity is equivalent to the apparently stronger mixing condition. We shall see later that much more than this is true: if α is ergodic then it automatically has the strongest of the whole known hierarchy of mixing properties.

The similarity of our definitions of ergodicity and mixing to each other enables us to prove both parts of the theorem simultaneously. Suppose that $f \in L_2(G)$ and λ, $|\lambda| = 1$, are such that $f^\alpha = \lambda f$.[8] We may expand f in a Fourier series in the characters $\phi \in G^*$, $f(x) = \sum_\phi a(\phi)\phi(x)$.[9] Concerning this series we must make two comments. First, even if G^*

[6] Hopf, op. cit., p. 9.

[7] Hopf, op. cit., p. 42.

[8] The fact that, considered as a transformation of L_2, α is a unitary operator implies that if there is any proper value λ at all then it must be of modulus one.

[9] Weil, op. cit., p. 76.

is uncountable, at most a countable number of ϕ's fail to be orthogonal to f, so that at most a countable number of a's are different from zero. Second, the series need converge only in the sense of L_2 (mean square convergence); the known fact that a sub-sequence of its partial sums converges almost everywhere is sufficient to justify the simple formal steps that follow. Replacing x by x^α we obtain

$$\sum_\phi \lambda a(\phi)\phi(x) = \lambda f(x) = f^\alpha(x) = f(x^\alpha)$$

$$= \sum_\phi a(\phi)\phi(x^\alpha) = \sum_\phi a(\phi)\phi^\alpha(x) = \sum_\phi a(\phi^{\alpha^{-1}})\phi(x).$$

Hence (using the orthogonality of the characters) we may equate coefficients and obtain $\lambda a(\phi) = a(\phi^{\alpha^{-1}})$, or $|a(\phi)| = |a(\phi^{\alpha^{-1}})|$. Since ϕ is arbitrary it follows that all coefficients $a(\phi)$ corresponding to ϕ's in the same orbit are equal in modulus. Since $\sum_\phi |a(\phi)|^2 < \infty$ it follows that all a's corresponding to ϕ's in the same infinite orbit vanish. This settles the *only if* part of our theorem: a non-constant f can exist only if α (considered as an automorphism of G^*) has finite orbits.

The converse is easier. Let $\phi \in G^*$ ($\phi \neq 1$) have a finite orbit; suppose, for definiteness, that n is the least positive integer for which $\phi^{\alpha^n} = \phi$. It follows that for the function $f = \phi + \phi^\alpha + \cdots + \phi^{\alpha^{n-1}}$ we have $f^\alpha = f$. The orthogonality and, a fortiori, linear independence of the ϕ's show that f is not constant. Since this shows that the existence of a finite orbit implies non-ergodicity, the proof of Theorem 1 is complete.

THEOREM 2. *Let α be any automorphism of the discrete abelian group H; if α has no finite orbits then it has an infinite number of orbits.*[10]

Case I. We assume first that there is in H an element ϕ_0 of finite order. By raising ϕ_0 to a suitable power we may assume that the order of ϕ_0 is a prime p. Write $\phi_n = \phi_0^{\alpha^n}$, $n = 0, \pm 1, \pm 2, \cdots$; we shall prove that the ϕ_n are independent mod p. (It is clear that the order of each ϕ_n is p.) Suppose, on the contrary, that $\phi_{i_1}^{r_1} \cdots \phi_{i_k}^{r_k} = 1$; it is merely a notational change to write $i_1 = 1, \cdots, i_k = k$, and we may, of course, assume that r_1 and r_k are not congruent to zero mod p. We have then

$$(1) \qquad \phi_1 = (\phi_2^{-r_2} \cdots \phi_k^{-r_k})^{r_1^{-1}}, \qquad \phi_k = (\phi_1^{-r_1} \cdots \phi_{k-1}^{-r_{k-1}})^{r_k^{-1}}.$$

(The exponents r_1^{-1}, r_k^{-1} make sense since we may interpret them in the modular field $GF(p)$.) Consider now the finite subgroup H_0 of H

[10] The author's thanks are due to R. Baer, R. H. Fox, and H. Samelson for many valuable discussions of this theorem and its proof.

generated by $\phi_1, \cdots, \phi_k$. It follows from the definition of the ϕ_n and the relations (1) that H_0 is invariant under both α and α^{-1} and contains consequently the entire orbit of ϕ_0. Since this contradicts the assumed nonexistence of finite orbits, the ϕ_n must indeed be independent. The desired conclusion follows at once: the elements $\psi_n = \phi_1 \cdots \phi_n$, $n = 1, 2, \cdots$, must all lie in different orbits.

Case II. If ϕ_0 is an element of infinite order we write $\phi_i = \phi_0^{\alpha^i}$ as before and we ask for which (positive or negative) integers i it is true that ϕ_i is a positive rational power of ϕ_0^r (that is, a positive integral power of some root of ϕ_0). If $\phi_i = \phi_0^r$ then $(\phi_{-i})^r = (\phi_0^r)^{\alpha^{-i}} = (\phi_i)^{\alpha^{-i}} = \phi_0$, so that ϕ_{-i} is an rth root of ϕ_0. If also $\phi_j = \phi_0^s$ then $\phi_{i+j} = (\phi_j)^{\alpha^i} = \zeta_0^{s\alpha^i} = \phi_0^{\alpha^i s} = \phi_0^{rs}$. In other words the set of i's under consideration is an additive group of integers; let i_0 be a generator of this group, $\phi_{i_0} = \zeta_0^{r_0}$. Then for any integer n

$$\phi_{i_0 n} = (\cdots((\phi_0)^{r_0})^{r_0}\cdots)^{r_0} = \phi_0^{r_0^n};$$

in other words the set of possible r's that may occur as exponents in a relation $\phi_i = \phi_0^r$ consists of all (positive, negative, or zero) integral powers of r_0. Hence the set of powers of ϕ_0 which are in the same orbit as ϕ_0 consists only of powers of r_0, and consequently there is a power of ϕ_0 which does not lie in this orbit. We may choose this power of ϕ_0 as a new starting element (of infinite order) and repeat the above argument ad infinitum. This completes the proof of Theorem 2.

We are now prepared to describe the spectral type of ergodic automorphisms. Let $\mathfrak{S}$ be a complete orthonormal set in an abstract, not necessarily separable, Hilbert space and denote by ψ any particular element of $\mathfrak{S}$. Arrange all remaining elements of $\mathfrak{S}$ as an infinite matrix in such a way that each row contains a countably infinite number of elements. Use as row index any set of suitable power and as column index the set of all (positive, negative, or zero) integers. A unique unitary operator σ is defined on Hilbert space by the requirement that it send ψ into itself and $\phi_{i,j}$ into $\phi_{i,j+1}$ for $j = 0, \pm 1, \pm 2, \cdots$, and all i. The spectral type of σ depends only on the number of rows; if we agree to use $\aleph$ for this cardinal number we may write $\sigma = \sigma(\aleph)$. We summarize our result in terms of these σ's.

THEOREM 3. *If α is a continuous ergodic automorphism of a compact abelian group G and $\aleph$ is the (necessarily infinite) cardinal number of G^* then α has the spectral type of the unitary operator $\sigma(\aleph)$.*

For the proof we need remember only that the characters of G form a complete orthonormal set of elements of $L_2(G)$. The principal character plays the role of ψ and the orbits of the other characters

may be written as the rows of the matrix mentioned above. Theorem 2 shows that there must be an infinite number of rows, and the well known fact that for any infinite cardinal $\aleph \cdot \aleph_0 = \aleph$[11] shows that the number of rows is the same as the total number of elements in G^*.[12]

3. Examples and questions. Let H be any compact abelian group and let G be the direct product of H with itself a countable number of times. We write G as the set of all sequences $\{x_n | n = 0, \pm 1, \pm 2, \cdots\}$, and we define a continuous ergodic automorphism α of G by the relations

$$\{x_n\}^\alpha = \{x_n'\}, \quad x_n' = x_{n+1}, \qquad n = 0, \pm 1, \pm 2, \cdots.$$

Transformations isomorphic to such α's (not only in the spectral but even in the stronger, measure theoretic, sense) were among the first known examples of ergodic transformations.

Examples apparently very simple from the algebraic point of view, but very difficult to handle geometrically, are furnished by the solenoids. Consider for instance the multiplicative group of all real numbers of the form e^r, where r is a dyadic rational number. The operation of squaring is an automorphism of this group (with no finite orbits), and hence an ergodic automorphism of its compact character group.

More in the classical spirit than either of the last two examples are the (continuous) automorphisms of the n-dimensional toral group. In order to retain our multiplicative notation we write the torus as n-tuples $(x_1, \cdots, x_n)$ of complex numbers of modulus one; thus the product of $(x_1, \cdots, x_n)$ by $(y_1, \cdots, y_n)$ is $(x_1y_1, \cdots, x_ny_n)$ and the identity element is $(1, \cdots, 1)$. It is well known that the automorphism group of the torus is the unimodular group, in the following sense. Given any $n \times n$ matrix $\alpha = (a_{ij})$ whose elements are integers and whose determinant is ± 1, we consider the mapping $(x_i) \to (\prod x_j^{a_{ij}})$: this mapping is the most general continuous automorphism of the torus. The condition of ergodicity—that is, the condition that, considered as an automorphism of the character group, α have no finite orbits—is equivalent in classical terms to the requirement that no root of unity should be a proper value of α. This remark enables us to write down any desired number of quite different looking analytic ergodic (and hence mixing) measure preserving transformations on the finite dimensional torus.

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[11] See F. Hausdorff, *Mengenlehre*, Berlin, 1935, p. 71.

[12] The first explicit discussion of the spectral form of a measure preserving transformation of type $\sigma(\aleph_0)$ was carried out by J. L. Doob and R. A. Leibler, *On the spectral analysis of a certain transformation*, Amer. J. Math. vol. 65 (1943) pp. 263–272.

This last example and the particular structure of the automorphisms it describes suggests a new, purely measure theoretic, invariant of measure preserving transformations. The Hamilton-Cayley equation says, in our notation, that if p is the characteristic polynomial of α then for every x in the torus $x^{p(\alpha)}=1$. It follows that for every character f of the torus we have, similarly, $f^{p(\alpha)}=1$. The existence or nonexistence of f's thus *annihilated* by certain polynomials in α (with, of course, integer coefficients) and, if they exist, their algebraic and measure theoretic structure, furnishes the invariant to which we referred. To illustrate the possible application of these invariants we mention the following: if it could be proved that the characters are (except for trivial changes on a set of measure zero) the only measurable functions of constant modulus one which are *annihilated* by $p(\alpha)$, it would follow rather easily that two ergodic automorphisms of the torus are measure theoretically isomorphic if and only if they correspond to conjugate elements in the unimodular group. We could thus obtain the first examples of measure theoretically distinct transformations of the spectral type of $\sigma(\aleph_0)$,[13] and the usual proper value theory would then point the way to further, more delicate, invariants.

In conclusion we mention an unsolved problem of purely technical interest, but one whose solution may throw some light on the deeper problems raised above. The question is simply: do there exist measure preserving transformations (on spaces of finite measure) of the spectral type of $\sigma(\aleph)$, where $\aleph$ is a *finite* cardinal?

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[13] The first examples of measure theoretically different but spectrally isomorphic transformations are due to J. von Neumann. These examples (not yet published) are, however, not mixing transformations.

CAUCHY-STIELTJES AND RIEMANN-STIELTJES INTEGRALS

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1. **Introduction.** This note treats the equivalence of the Riemann-Stieltjes and Cauchy-Stieltjes integrals (abbreviated RS and CS integrals) and conditions for the existence and equality of the latter. The ordinary Riemann, the left Cauchy, and the right Cauchy integrals are defined as limits of the sums $\sum_1^n f(\xi_i)(x_i - x_{i-1})$, $x_{i-1} \leqq \xi_i \leqq x_i$, $\sum_1^n f(x_{i-1})(x_i - x_{i-1})$ and $\sum_1^n f(x_i)(x_i - x_{i-1})$ respectively; it is known [2] that these integrals are equivalent. Corresponding to these integrals, we have the RS and the two CS integrals, defined as limits of the sums $\sum_1^n f(\xi_i)[g(x_i) - g(x_{i-1})]$, $\sum_1^n f(x_{i-1})[g(x_i) - g(x_{i-1})]$, and $\sum_1^n f(x_i)[g(x_i) - g(x_{i-1})]$. The right modified RS integral is obtained from the sums $\sum_1^n f(\xi_i)[g(x_i) - g(x_{i-1})]$, $x_{i-1} \leqq \xi_i < x_i$. Examples in §4 show that the CS integrals may exist, with equal or unequal values, when the RS and the right modified RS do not; that the right modified RS integral may exist when the RS does not; and that one of the CS integrals may exist when the other does not. Thus the RS, the right modified RS, and the two CS integrals are not equivalent. Since all these integrals obviously exist when the RS integral does, it is natural to investigate conditions under which the existence of a CS or right modified RS integral implies the existence of the RS integral. It is shown in this note that if g is non-decreasing, if f and g have no common discontinuities on the same side, and if the left CS integral exists, then the RS integral exists and has the same value, the integrals being limits in the sense of increasing refinement of subdivisions. This result is established in two steps: (a) if g is non-decreasing, if f and g have no common discontinuities on the right, and if the left CS integral exists, then the right modified RS integral exists; (b) if the right modified RS integral exists, if g is non-decreasing, and if f and g have no common discontinuities on the left, then the RS integral exists. This result obviously includes the previously proved equivalence of the Riemann and Cauchy integrals [2] and certain others [3]. Further, it states sufficient conditions for the equality of the two CS integrals; these conditions show that the two ordinary Cauchy integrals are always equal. The note closes with a proof that the CS integrals exist when f has only simple discontinuities and g has bounded variation. We conclude from this result and others stated above that both of the CS integrals properly include the RS integral, and that neither CS integral includes the other. Precise statements of

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theorems are given in §5.

2. **Notation.** Let f and g denote real, single valued, bounded functions defined on $a \leqq x \leqq b$. Let X denote a subdivision $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $a \leqq x \leqq b$, and let $x' \leqq x \leqq x''$ denote a generic interval of X. Let $X_2 \geqq X_1$ denote that X_2 is a refinement of X_1, and let X_1X_2 denote the subdivision determined by all the points of division of X_1 and X_2. Also let $\|X\|$ denote the norm of X, that is, the maximum length of a subinterval of X. Finally, let $X\Psi(\Delta)$ denote the sum of the values of the interval function Ψ for all the intervals Δ of X. The interval functions to be met below are of the form $f(\xi)[g(x'') - g(x')]$, $x' \leqq \xi \leqq x''$. This notation has been used by Geöcze [1].

3. **Definitions.** In this section we give definitions of various integrals.

DEFINITION. *We say f has a left* CS X *integral with respect to g, denoted by $(C_L, X)\int_a^b f dg$, if and only if for each $\epsilon > 0$ there exists an $X_0(\epsilon)$ such that*

$$(1) \qquad \left| (C_L, X)\int_a^b f dg - Xf(x')[g(x'') - g(x')] \right| < \epsilon, \qquad X \geqq X_0(\epsilon).$$

DEFINITION. *We say f has a left* CS $\|X\|$ *integral with respect to g, denoted by $(C_L, \|X\|)\int_a^b f dg$, if and only if for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that*

$$(2) \qquad \left| (C_L, \|X\|)\int_a^b f dg - Xf(x')[g(x'') - g(x')] \right| < \epsilon, \qquad \|X\| \leqq \delta(\epsilon).$$

The definitions of the corresponding right CS integrals $(C_R, X)\int_a^b f dg$, $(C_R, \|X\|)\int_a^b f dg$ are obtained by replacing $f(x')$ by $f(x'')$ in (1) and (2). Also, if $f(x')$ is replaced by $f(\xi)$, $x' \leqq \xi \leqq x''$, we obtain the RS integrals $(R, X)\int_a^b f dg$, $(R, \|X\|)\int_a^b f dg$. If $f(x')$ in (1) is replaced by $f(\xi)$, $x' \leqq \xi < x''$, we have the right modified RS X integral $(R^*, X)\int_a^b f dg$.

4. **Examples.** A first example shows that the CS integrals may exist, with equal or unequal values, even when the RS and the right modified RS integrals do not, and that the right modified RS integral may exist when the RS does not. Let f have the value 1 when $x = 0$ and the value 0 elsewhere on the interval $-1 \leqq x \leqq 1$. Set $g(0) = c$, $g(x) = -1$ on $-1 \leqq x < 0$, and $g(x) = 1$ on $0 < x \leqq 1$. Then $(C_L, X)\int_{-1}^1 f dg = 1 - c$, $(C_R, X)\int_{-1}^1 f dg = 1 + c$. The $\|X\|$ CS integrals do not exist, and neither do the X and $\|X\|$ RS integrals. If $c = 1$, then $(R^*, X)\int_{-1}^1 f dg$

exists and equals $(C_L, X)\int_{-1}^{1} f dg$.

A second example shows that one CS integral may exist when the other does not. Let $f(x)=\sin(1/x)$ and $g(x)=x-1$ on $-1\leqq x<0$; $f(x)=0$ and $g(x)=x+1$ on $0\leqq x\leqq 1$. Then $(C_R, X)\int_{-1}^{1} f dg$ exists but $(C_L, X)\int_{-1}^{1} f dg$ does not (see a necessary condition for the existence of the latter [3, p. 267, 2.14]).

5. **Theorems.** The results to be established will be stated in this section.

THEOREM 1. *Let f and g, defined on $a\leqq x\leqq b$, have the following properties*:

(i) $|f(x)|\leqq M$;

(ii) $g(x)$ *is non-decreasing*;

(iii) f *and* g *have no common discontinuities on the left*;

(iv) $(R^*, X)\int_a^b f dg$ *exists.*

Then $(R, X)\int_a^b f dg$ *exists and equals* $(R^*, X)\int_a^b f dg$.

THEOREM 2. *Let f and g satisfy* (i), (ii) *and the following*:

(v) f *and* g *have no common discontinuities on the right*;

(vi) $(C_L, X)\int_a^b f dg$ *exists.*

Then $(R^*, X)\int_a^b f dg$ *exists and equals* $(C_L, X)\int_a^b f dg$.

COROLLARY 1. *Let f and g satisfy* (i), (ii), (vi), *and the following*:

(vii) f *and* g *have no common discontinuities on the same side.*

Then $(R, X)\int_a^b f dg$, $(C_R, X)\int_a^b f dg$ *exist and equal* $(C_L, X)\int_a^b f dg$.

COROLLARY 2. *If the hypotheses of Corollary* 1 *are strengthened by replacing* (vii) *by*

(viii) f *and* g *have no common discontinuities,*

then $(R, \|X\|)\int_a^b f dg$ *also exists.*

THEOREM 3. *If f satisfies* (i) *and has only simple discontinuities, and if g has bounded variation, then* $(C_L, X)\int_a^b f dg$ *and* $(C_R, X)\int_a^b f dg$ *exist.*

6. **Proofs of the theorems.** The proof of Theorem 1 will be omitted; a more general result is known [3, p. 273].

Consider Theorem 2. Let ϵ be an arbitrary positive constant, and let $X_0(\epsilon)$ be the subdivision which exists by (1) and (vi). Let η be another arbitrary positive number. In each interval $x'\leqq x\leqq x''$ of X_0 define a point t as follows: (a) $t=x''$ if $f(x)\geqq f(x')+\eta$ has no solution x satisfying $x'<x<x''$; (b) $t=$g.l.b. $E_x[f(x)\geqq f(x')+\eta, x'<x<x'']$ in case this set is not empty. With each point t associate a sequence of points $t_1, t_2, \cdots$ as follows: (a) $t_k=t$, $k=1, 2, \cdots$, if $t=x''$; (b) $t_k=t$, $k=1, 2, \cdots$, if $f(t)\geqq f(x')+\eta$; (c) in the remaining case

$\{t_k\}$ is a sequence such that $f(t_k) \geqq f(x') + \eta$, $k = 1, 2, \cdots$, and $\lim_k t_k = t$. The third case occurs only when f is discontinuous on the right at t; at such a point g is continuous on the right by (v). Then for every interval of X_0 $\lim_k [g(x'') - g(t_k)] = [g(x'') - g(t)]$. This fact will be used to prove

$$X_0\eta[g(x'') - g(t)] \leqq 2\epsilon. \tag{3}$$

Let X_1 be the subdivision obtained by adding the points of division t to those of X_0, and let X_1^k be the subdivision obtained from X_0 by subdividing each of its intervals by t_k. Since $X_1^k \geqq X_0$, we have

$$| X_1^k f(x')[g(x'') - g(x')] - X_0 f(x')[g(x'') - g(x')] | < 2\epsilon \tag{4}$$

by (1) and (vi). But since

$$X_1^k f(x')[g(x'') - g(x')] = X_0 f(x')[g(t_k) - g(x')] + X_0 f(t_k)[g(x'') - g(t_k)],$$

we find that the left-hand side of (4) reduces to

$$| X_0[f(t_k) - f(x')][g(x'') - g(t_k)] | \geqq X_0\eta[g(x'') - g(t_k)]. \tag{5}$$

Thus from (4) and (5) we find $X_0\eta[g(x'') - g(t_k)] < 2\epsilon$; by taking the limit as $k \to \infty$, we obtain (3).

Let $U^*(f; x', x'')$ and $L^*(f; x', x'')$ denote the l.u.b. and g.l.b. respectively of f on $x' \leqq x < x''$. Then since

$$\begin{aligned} &X_1U^*(f; x', x'')[g(x'') - g(x')] \\ &\quad = X_0U^*(f; x', t)[g(t) - g(x')] + X_0U^*(f; t, x'')[g(x'') - g(t)], \end{aligned}$$

we have

$$\begin{aligned} &| X_1U^*(f; x', x'')[g(x'') - g(x')] - X_0 f(x')[g(x'') - g(x')] | \\ &\quad \leqq | X_0[U^*(f; x', t) - f(x')][g(t) - g(x')] | \\ &\qquad + | X_0[U^*(f; t, x'') - f(x')][g(x'') - g(t)] | \\ &\quad \leqq \eta[g(b) - g(a)] + 2MX_0[g(x'') - g(t)] \leqq \eta[g(b) - g(a)] + 4M\epsilon/\eta \end{aligned}$$

by (i), (ii), (3), and the definition of t. Thus

$$\begin{aligned} &\left| X_1U^*(f; x', x'')[g(x'') - g(x')] - (C_L, X)\int_a^b f dg \right| \\ &\qquad \leqq \eta[g(b) - g(a)] + 4M\epsilon/\eta + \epsilon \end{aligned}$$

by the result just established and the definition of X_0. If ζ is an arbitrary positive number, it is possible to choose η and ϵ so small that

the expression on the right of the last inequality is less than ζ. By a similar argument we can show that there exists a subdivision X_2 such that

$$\left| X_2 L^*(f; x', x'')[g(x'') - g(x')] - (C_L, X)\int_a^b f dg \right| < \zeta.$$

It follows from these results that

$$\left| Xf(\xi)[g(x'') - g(x')] - (C_L, X)\int_a^b f dg \right| < \zeta$$

for $x' \leqq \xi < x''$ and any $X \geqq X_1 X_2$. Thus $(R^*, X)\int_a^b f dg$ exists and equals $(C_L, X)\int_a^b f dg$, and the proof of Theorem 2 is complete.

Corollary 1 follows from Theorems 1 and 2. A sufficient condition that $(R, \|X\|)\int_a^b f dg$ exist is that $(R, X)\int_a^b f dg$ exist, and that f and g have no common discontinuities [3, p. 269, 4.14]; Corollary 2 follows from this condition and Corollary 1.

Before we begin the proof of Theorem 3, two preliminary results are needed. Let g be any function of bounded variation on $a \leqq x \leqq b$, and let x' be any point such that $a \leqq x' < b$. Then given any $\zeta > 0$, there exists an x'' sufficiently near x' on the right so that the total variation of g on any interval $t \leqq x \leqq x''$, where $x' < t < x''$, is less than ζ. To prove this statement we define $h(x) = g(x)$, $x \neq x'$, and $h(x') = g(x' + 0)$ and recall that the total variation of $h(x)$ is continuous on the right at x' [6, p. 356]. The second preliminary result is the following. If f is defined on $\alpha \leqq x \leqq \beta$ and has simple discontinuities only, and if the oscillation of f at each point of the interval $\alpha \leqq x < \beta$ is less than λ, then $\alpha \leqq x \leqq \beta$ can be subdivided into a finite number of intervals $x' \leqq x \leqq x''$ such that the oscillation of f on $x' \leqq x < x''$ is less than 2λ. The proof follows readily from a theorem of Baire [4, p. 311].

Finally, consider the proof of Theorem 3. Let any $\epsilon > 0$ be given. Form a subdivision $X_0(\epsilon)$ of $a \leqq x \leqq b$ by points $a = x_0 < x_1 < \cdots < x_n = b$ so that (a) each of the r points at which the oscillation of f is equal to or greater than λ is a point of division; (b) if $x' \leqq x \leqq x''$ is an interval of X_0 such that the oscillation of f at x' is equal to or greater than λ, the total variation of g on any interval $t \leqq x \leqq x''$, $x' < t < x''$, is less than ζ; (c) if $x' \leqq x \leqq x''$ is not an interval of the type described in (b), the oscillation of f on $x' \leqq x < x''$ is less than 2λ.

Let X_1 be any refinement of X_0. A straightforward calculation shows that

$$(6)\quad \left| X_1 f(x')[g(x'') - g(x')] - X_0 f(x')[g(x'') - g(x')] \right| < 2Mr\zeta + 2\lambda V,$$

where V is the total variation of g on $a \leqq x \leqq b$. We may suppose that

λ was chosen so that $2\lambda V < \epsilon/4$. This choice of λ determines the integer r. We may suppose further that ζ was chosen so that $2Mr\zeta < \epsilon/4$. Then the right-hand member of (6) is less than $\epsilon/2$. Similarly, for $X_2 \geqq X_0$, we have

$$| X_0 f(x')[g(x'') - g(x')] - X_2 f(x')[g(x'') - g(x')] | < \epsilon/2;$$

hence

$$| X_1 f(x')[g(x'') - g(x')] - X_2 f(x')[g(x'') - g(x')] | < \epsilon,$$
$$X_1, X_2 \geqq X_0(\epsilon).$$

This is a sufficient condition that $(C_L, X)\int_a^b f dg$ exists [5, p. 106]. A similar proof shows that $(C_R, X)\int_a^b f dg$ also exists.

All proofs are now complete.

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EXACT nTH DERIVATIVES

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Let y be a function of x with derivatives of all orders, and let θ be a function of x, y, and a finite number of derivatives of y. If, independently of the choice of the function y, θ is the nth total derivative of some function ψ of x, y, and derivatives of y, then we shall call θ an *exact nth derivative*. The problem with which this note is concerned is to determine, for any given function θ and positive integer n, if θ is an exact nth derivative. The case for which $n=1$ is completely covered by the well known Euler differential equation which arises in the simplest problem of the calculus of variations. For a function θ to be an exact first derivative, it is necessary and sufficient that θ satisfy the Euler differential equation. The contribution of this paper is the treatment of the cases in which n exceeds unity. A system of n differential equations is developed, satisfaction of which by θ constitutes a necessary condition that θ be an exact nth derivative. These equations do not yield an altogether satisfactory sufficient condition. It turns out that if θ satisfies the equations in question, it may still fail to be an exact nth derivative. However, under these circumstances, θ must differ from an exact nth derivative by a function of very special character.

Notation. Let us suppose y to be an arbitrary function of x possessing derivatives of all orders. We shall denote the jth derivative of y with respect to x by y_j, and sometimes denote y itself by y_0. We suppose θ to be a function of x, y, and of finitely many of the y_j, possessing partial derivatives of all orders with respect to all its arguments. The operation of differentiation with respect to x will be indicated by the symbol D; thus $D=\partial/\partial x+\sum y_{i+1}\partial/\partial y_i$. We shall understand that the range of the subscript i in D extends from zero to plus infinity, recognizing that when D operates on a function of x, y, and of finitely many of the y_j it reduces to a finite sum. The symbol D^i, where i is a positive integer, will denote the operation of taking the ith derivative. We shall use the expression $C_{p,q}$ to denote the binomial coefficient $p\cdot(p-1)\cdot\cdot\cdot(p-q+1)/q!$ where q is a non-negative integer and p is any integer.

Summary of results. Let n be a positive integer. Let operators E_t, $t=1, \cdots, n$, be defined as follows. Expand, formally, $E_t=(1+D\partial/\partial y_1)^{-t}\partial/\partial y$ as the product by $\partial/\partial y$ of a power series in $D\partial/\partial y_1$, and replace terms $(D\partial/\partial y_1)^i\partial/\partial y$ by $D^i\partial/\partial y_i$. *Let there be a*

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function ψ of x, y, and of finitely many y_i such that $D^n\psi=\theta$ identically in x and y. Then[1]

$$E_t\theta = 0, \qquad t = 1, 2, \cdots, n, \tag{1}$$

identically in x and y.

We also establish a partial converse to this result. *Let θ satisfy equations* (1) *identically in x and y. Then there exist polynomials in x $\pi_1, \pi_2, \cdots, \pi_{n-1}$, where the degree of each nonzero π_i is less than i, such that $\theta-\pi_1y_1-\cdots-\pi_{n-1}y_{n-1}$ is an exact nth derivative.* Note that no term in y_0 need be subtracted from θ to render θ exact, so that for n equal to unity we see that θ itself is a first derivative if it satisfies (1). The equation $E_1\theta=0$ is simply the Euler differential equation, so that for n equal to unity our result restates the well known property of the Euler equation.

Necessity proof. We use the relations

$$\frac{\partial}{\partial y_r}D^n = \sum_{i=0}^{\min(r,n)} C_{n,i}D^{n-i}\frac{\partial}{\partial y_{r-i}}, \tag{2}$$

$$r = 0, 1, 2, \cdots; n = 1, 2, 3, \cdots,$$

which we proceed to establish. For r equal to zero, this amounts to the statement that $(\partial/\partial y)D^n=D^n\partial/\partial y$, $n=1, 2, 3, \cdots$. Clearly

$$\frac{\partial}{\partial y}D = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}+\sum y_{i+1}\frac{\partial}{\partial y_i}\right) = \frac{\partial^2}{\partial y\partial x}+\sum y_{i+1}\frac{\partial^2}{\partial y\partial y_i}.$$

Since the order in which the partial derivatives are extracted can be reversed, this last expression equals $D\partial/\partial y$. This disposes of the case $r=0$, $n=1$, and the cases $r=0$, n arbitrary, follow immediately.

We now treat the cases in which r is greater than zero. In evaluating $(\partial/\partial y_r)D$ we must consider that y_r appears in the term $y_r\partial/\partial y_{r-1}$ as a coefficient of a partial derivative, which was not the case for r equal to zero. We obtain then

$$\frac{\partial}{\partial y_r}D = D\frac{\partial}{\partial y_r}+\frac{\partial}{\partial y_{r-1}}.$$

Consequently,

$$\frac{\partial}{\partial y_r}D^n = D\frac{\partial}{\partial y_r}D^{n-1}+\frac{\partial}{\partial y_{r-1}}D^{n-1}.$$

Proceeding by induction, we assume that the right member of this

[1] These equations are written out explicitly as equations (4) below.

equation has a representation

$$D \sum_{i=0}^{\min(r,n-1)} C_{n-1,i} D^{n-1-i} \frac{\partial}{\partial y_{r-i}} + \sum_{i=0}^{\min(r-1,n-1)} C_{n-1,i} D^{n-1-i} \frac{\partial}{\partial y_{r-1-i}}.$$

Collecting similar terms, we see that the coefficient of $D^{n-i}\partial/\partial y_{r-i}$ is $C_{n-1,i}+C_{n-1,i-1}$. It is well known that this sum is $C_{n,i}$. This suffices to prove that if relations (2) hold for values less than n and less than or equal to r, they likewise hold for n and r. It remains to discuss the transition from n and $r-1$ to n and r. This part of the proof is so similar to the foregoing discussion that the details may be omitted.

From now on n is to be a positive integer, arbitrary but fixed.

We multiply both members of the rth of equations (2) by D^r, $r=1, 2, \cdots$. As a consequence, the power of D which multiplies any given $\partial/\partial y_s$ is the same in all the modified equations (2) in which the expression appears, namely the $(n+s)$th power.

Denote the terms $D^r(\partial/\partial y_r)D^n$ by Λ_r, $r=0, 1, 2, \cdots$. After the above multiplication, equations (2) furnish us expressions for the Λ_r as linear combinations, with numerical coefficients, of terms $D^{n+s}\partial/\partial y_s$. The numerical coefficients can be readily obtained. A term $D^{n+s}\partial/\partial y_s$ appears first in the expression for Λ_s, and is present, in all, in $\Lambda_s, \Lambda_{s+1}, \cdots, \Lambda_{s+n}$. Its numerical coefficient in each of these Λ_{s+i} is $C_{n,i}$.

We now seek numbers $a_0, a_1, a_2, \cdots$, such that $\sum_{i=0}^{\infty} a_i\Lambda_i$ is identically zero; that is, in this sum the numerical coefficient of each term $D^{n+s}\partial/\partial y_s$ is to be zero. This implies that the a_i must satisfy

$$\sum_{i=0}^{n} a_{r+i} C_{n,i} = 0, \qquad r = 0, 1, 2, \cdots. \tag{3}$$

In order to solve these equations with a minimum of computation, we formulate an equivalent problem involving power series. Let u be an indeterminate and let $\sum a_i u^i$ be a formal power series in u, with numerical coefficients. Observe that in the product $(1+u)^n \sum a_i u^i$ the coefficient of u^{n+r}, $r=0, 1, 2, \cdots$, is precisely the left member of the corresponding equation (3). Thus if we can attribute values to the a_i which make the above product a polynomial of degree less than n, these a_i will also be solutions of (3). But the solution of this problem is immediate. Expand any one of the fractions $1/(1+u)^t$, $t=1, \cdots, n$, as a power series in u. The coefficients obtained in this way obviously serve as solutions for the polynomial problem, and, consequently, furnish solutions for equations (3).

The proof can now be quickly completed. Let θ be the nth derivative

of a function ψ; in symbols $\theta = D^n\psi$. Then $D^r\partial\theta/\partial y_r = \Lambda_r\psi$. Since $\sum a_r\Lambda_r = 0$ we have $\sum a_r D^r\partial\theta/\partial y_r = 0$. Using explicit values for the a_r we have

$$\frac{\partial\theta}{\partial y} + \sum_{r=1}^{\infty} C_{-t,r} D^r \frac{\partial\theta}{\partial y_r} = 0, \qquad t = 1, 2, \cdots, n. \tag{4}$$

The details of the proof make plain the connection between equations (4) and equations (1).

Sufficiency proof. We need the following lemma.

Let n be a positive integer and let $\pi(u)$ be a polynomial in u, not identically zero, whose degree is less than n. Then in the expansion of the function $\mu(u) = \pi(u)/(1+u)^n$ as a power series in u the coefficients of no n consecutive powers of u are zero.

This is obviously true for n equal to unity, for the series of $1/(1+u)$ has no missing terms, and multiplication by a nonzero constant cannot introduce any such terms. We prove the lemma by induction on n. Representing $\pi(u)$ as a polynomial in $(1+u)$, we have $\pi(u) = b_0 + b_1(1+u) + \cdots + b_{n-2}(1+u)^{n-2} + b_{n-1}(1+u)^{n-1}$ where not all the b_i are zero. After being multiplied by $(1+u)$ our function has the form

$$(1+u)\mu(u) = [b_0 + b_1(1+u) + \cdots + b_{n-2}(1+u)^{n-2}]/(1+u)^{n-1} + b_{n-1}. \tag{5}$$

The number of missing terms in $\mu(u)$ is related in a simple way to the number of missing terms in the product of this function by $(1+u)$. We distinguish two cases. If the first n coefficients of $\mu(u)$ are zero, then they are likewise zero in the product. If, on the other hand, at least n consecutive coefficients in $\mu(u)$ are zero, and if furthermore this gap is preceded by nonzero terms, then in the product at least $n-1$ consecutive coefficients are zero and this gap must also be preceded by nonzero terms. We show that each of these cases leads to a contradiction. We may suppose that not all of $b_0, b_1, \cdots, b_{n-2}$ are zero, for otherwise the problem reduces to the case of n equal to unity. Thus we see from (5) that $(1+u)\mu(u)$ is the sum of a constant and a function which is subject to our induction hypotheses. This function can have at most $n-2$ consecutive missing terms. We have seen that if $\mu(u)$ is to lack n consecutive terms, $(1+u)\mu(u)$ must lack at least $n-1$ such terms. We have also seen that in (5) the first summand can have at most $n-2$ missing terms. We conclude that it is the addition of b_{n-1} to this summand which contributes another zero coefficient. But this can only be the case if it is the first $n-1$ terms of

$(1+u)\mu(u)$ which are zero. In this case our argument shows that if $\mu(u)$ lacked n consecutive terms, $(1+u)\mu(u)$ would also have n consecutive missing terms. This discrepancy proves the lemma.

We are now in a position to prove our partial converse to the result established above. Let θ be a function of x, y, and the y_j, such that the relations $E_t\theta=0$, $t=1, 2, \cdots, n$, are identities in x and y. If the partial derivatives of θ with respect to all the y_j vanish, then θ is a function of x alone, and is certainly an exact nth derivative. We assume henceforth that not all these partial derivatives of θ vanish.

Let m be the largest integer for which $\partial\theta/\partial y_m \neq 0$. Assuming that m is not less than n, we are going to show that there exists a function ψ such that $\theta - D^n\psi$ contains effectively no derivative y_j of order greater than $n-1$. For this purpose we prove first that θ is linear in y_m, and that in θ the coefficient $\partial\theta/\partial y_m$ of y_m is free of $y_{m-1}, y_{m-2}, \cdots, y_{m-n+1}$.

It follows from the theory of determinants that there exist constants $b_1, b_2, \cdots, b_n$, not all zero, such that $b_1E_1+b_2E_2+\cdots+b_nE_n$ is free of $D^i\partial/\partial y_i$ where i has values from $m-n+1$ to $m-1$ inclusive. By virtue of our lemma, this linear combination of the E_i must contain effectively the expression $D^m\partial/\partial y_m$. We suppose the b_i selected in such a way that the coefficient of this expression is unity. Then, because θ does not contain a y_s with s greater than m, we have

$$(6) \qquad \sum_{i=1}^{n} b_iE_i\theta = D^m\frac{\partial\theta}{\partial y_m} + cD^{m-n}\frac{\partial\theta}{\partial y_{m-n}} + \cdots + g\frac{\partial\theta}{\partial y},$$

where $c, \cdots, g$ are constants. The operators E_i are linear, so our assumption that θ is annulled by the E_i implies that both members of (6) are zero. If the term $D^m\partial\theta/\partial y_m$ is zero, it must be that $\partial\theta/\partial y_m$ is a polynomial in x of degree less than m. This certainly agrees with our statement concerning the manner in which y_m is present in θ. Assuming now that this term is not zero, we show that $\partial\theta/\partial y_m$ is free of $y_m, y_{m-1}, \cdots, y_{m-n+1}$. Suppose this is not so. Let y_r be the derivative of greatest order effectively present in $\partial\theta/\partial y_m$. Then y_{m+r} is effectively present in $D^m\partial\theta/\partial y_m$. The other terms of the right member of (6) cannot contain effectively derivatives of y of order greater than $2m-n$, because the partial derivatives of θ are of order at most m and the differentiations increase this order by at most $m-n$. Thus $m+r$ does not exceed $2m-n$, whence r does not exceed $m-n$. This shows that θ differs from $y_m\partial\theta/\partial y_m$ by a function of order less than m, and that $\partial\theta/\partial y_m$ is free of $y_m, y_{m-1}, \cdots, y_{m-n+1}$.

We now introduce an auxiliary function ψ_m described by the relation

$$\psi_m = \int \frac{\partial \theta}{\partial y_m} dy_{m-n},$$

where the integration is performed with respect to y_{m-n}, treating $x, y, y_1, \cdots, y_{m-n}$ as independent variables. Because of the use of the indefinite integral in its definition, ψ_m contains an arbitrary additive function of $x, y, y_1, \cdots, y_{m-n-1}$; thus the integration actually leads to many functions, some of which may not possess partial derivatives of all orders. However, the integration also yields functions with partial derivatives of all orders, and it is from these that we suppose a definite ψ_m to be selected. This function also has the property that its partial derivative with respect to y_{m-n} is $\partial\theta/\partial y_m$. Because the order of $\partial\theta/\partial y_m$ is less than $m-n+1$ it follows that $D\psi_m$ differs from $y_{m-n+1}\partial\theta/\partial y_m$ by a function of order less than $m-n+1$. Similarly $D^n\psi_m$ differs from $y_m\partial\theta/\partial y_m$ by a function of order less than m. Then the order of $\theta - D^n\psi_m$ is less than m. In addition, this difference is annulled by all the operators $E_1, \cdots, E_n$. If the order of $\theta - D^n\psi_m$ exceeds $n-1$ we can follow the same procedure with this new function subtracting from it an exact nth derivative and reducing its order still further. After a finite number of steps we obtain a function ψ such that the order of $\theta - D^n\psi$ is less than n.

Let us denote the difference $\theta - D^n\psi$ by τ. It may be that τ is identically zero. In this case θ is an exact nth derivative. Even if τ is not zero, it must still be annulled by $E_1, \cdots, E_n$. Because the order of τ is less than n, the equations $E_i\tau = 0$, $i = 1, 2, \cdots, n$, constitute a system of n homogeneous linear equations for the n quantities $\partial\tau/\partial y, D\partial\tau/\partial y_1, \cdots, D^{n-1}\partial\tau/\partial y_{n-1}$. It follows from the lemma that the determinant of this system is not zero. We conclude that each of the quantities $\partial\tau/\partial y, D\partial\tau/\partial y_1, \cdots, D^{n-1}\partial\tau/\partial y_{n-1}$ is zero. Thus τ is a function of the form $\pi_1 y_1 + \pi_2 y_2 + \cdots + \pi_{n-1} y_{n-1}$ where each π_i is a polynomial in x of degree less than i. We do not overlook the fact that τ could also contain a term free of the y_j, but since such a term, being a function of x alone, would be an exact nth derivative, we may suppose it incorporated into ψ. This completes the proof of the results enunciated at the beginning of the paper.

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